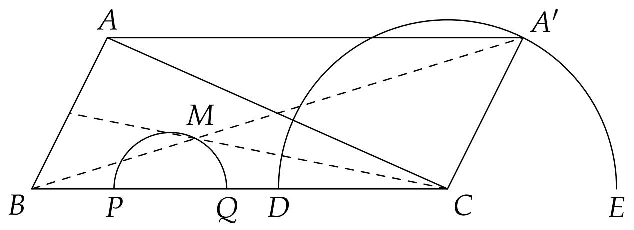
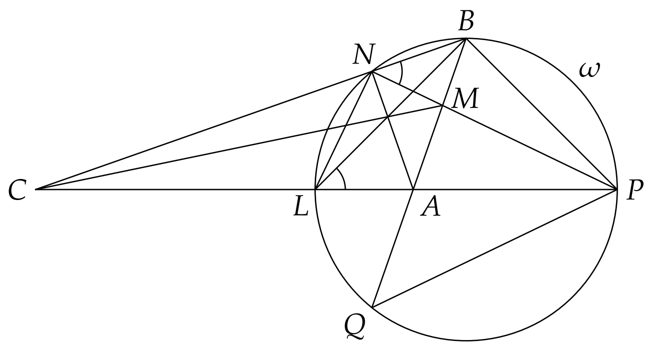
Baltic Way

2002–2006

Problems and Solutions





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Baltic Way 2002–2006 — Problems and Solutions

Collected and edited by Rasmus Villemoes.

This booklet contains the problems from the Baltic Way competitions held in the period 2002–2006, together with suggested solutions. The material may freely be used for educational purposes; however, if part of this booklet is published in any form, it is requested that the Baltic Way competions are cited as the source.

Typeset using the LaTeX document preparation system and the memoir document class. All figures are drawn using METAPOST.

An electronic version of this booklet is available at the web site of the Baltic Way 2007 competition:

www.balticway07.dk

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FOREWORD

The Baltic Way is a competition in mathematics for students in the secondary school. It was held for the first time in 1990 with participants from Estonia, Latvia and Lithuania. It is named after a mass demonstration in August 1989, when about two million inhabitants of these countries stood holding hands along the road from Tallinn to Vilnius on the 50th anniversary of the Molotov-Ribbentrop Pact. After the political changes which were soon to follow, it became possible to invite teams from other countries, making the competition a truly international event. Since 1990, it has taken place each year, hosted by different countries, and since 1997, teams from all countries around the Baltic Sea plus the Nordic countries Norway and Iceland have participated. Iceland, which was the first country to recognise the independence of the Baltic states, was actually one of the first other than the three Baltic ones to be invited to send a team. Occasionally, teams from countries outside the aforementioned permanent group have participated as guests.

Each team consists of five students, who are requested to collaborate on the solution of a set of twenty problems to be answered within four and a half hours. Just one solution of each problem is accepted from each team. Thus it is a part of the challenge given to the teams that they must collaborate to utilise optimally the individual skills of their members.

This booklet contains the problems of the Baltic Way competitions 2002–2006 with suggested solutions and tables of the scores of the teams. A printed copy is given to participants in the Baltic Way 2007 in Copenhagen, and a PDF version is available at www.balticway07.dk/earlierbw.php. It is based on collections of the problems of each year with suggested solutions which were produced by the respective organisers and published at their web sites. I should like to thank the authors of this material for providing me with their original TEX-files.

Previously, Marcus Better compiled the problems from the years 1990–1996, and Uve Nummert and Jan Willemson the problems from 1997–2001, with suggested solutions in printed booklets. None of these booklets are available by now. The problems from all the years are posted with suggested solutions at several web sites. An almost complete collection is found at the web site of the Baltic Way 2007 in Copenhagen, www.balticway07.dk.

A great effort has been made to ensure that the provided solutions are complete and accurate. However, should you encounter an error or a typo, please write to me using the email address below. Of course, other comments are also most welcome.

Hopefully, this collection of problems of varying difficulty will be useful in education and training for national and international competitions in mathematics. Everyone is free to use it for such purposes. If the problems, or some of them, are published in print or on the World Wide Web, a due citation in the form "Baltic Way 200x" is requested.

Århus, October 2007

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PROBLEMS

Baltic Way 2002

Tartu, Estonia

October 31–November 4, 2002

02.1. (p. 15) Solve the system of equations

$$a^3 + 3ab^2 + 3ac^2 - 6abc = 1$$
$$b^3 + 3ba^2 + 3bc^2 - 6abc = 1$$
$$c^3 + 3ca^2 + 3cb^2 - 6abc = 1$$

in real numbers.

02.2. (p. 15) Let a, b, c, d be real numbers such that

$$a + b + c + d = -2,$$
$$ab + ac + ad + bc + bd + cd = 0.$$

Prove that at least one of the numbers a, b, c, d is not greater than -1.

02.3. (p. 16) Find all sequences $a_0 \le a_1 \le a_2 \le \cdots$ of real numbers such that

$$a_{m^2+n^2} = a_m^2 + a_n^2$$

for all integers $m, n \ge 0$.

02.4. (p. 17) Let n be a positive integer. Prove that

$$\sum_{i=1}^{n} x_i(1 - x_i)^2 \le \left(1 - \tfrac{1}{n}\right)^2$$

for all nonnegative real numbers $x_1, x_2, \ldots, x_n$ such that $x_1 + x_2 + \cdots + x_n = 1$.

02.5. (p. 18) Find all pairs (a, b) of positive rational numbers such that

$$\sqrt{a} + \sqrt{b} = \sqrt{2 + \sqrt{3}}.$$

02.6. (p. 18) The following solitaire game is played on an $m \times n$ rectangular board, $m, n \ge 2$, divided into unit squares. First, a rook is placed on some square. At each move, the rook can be moved an arbitrary number of squares horizontally or vertically, with the extra condition that each move has to be made in the 90° clockwise direction compared to the previous one (e.g. after a move to the left, the next one has to be done upwards, the next one to the right etc.). For which values of m and n is it possible that the rook visits every square of the board exactly once and returns to the first square? (The rook is considered to visit only those squares it stops on, and not the ones it steps over.)

02.7. (p. 19) We draw n convex quadrilaterals in the plane. They divide the plane into regions (one of the regions is infinite). Determine the maximal possible number of these regions.

02.8. (p. 19) Let P be a set of $n \ge 3$ points in the plane, no three of which are on a line. How many possibilities are there to choose a set T of $\binom{n-1}{2}$ triangles, whose vertices are all in P, such that each triangle in T has a side that is not a side of any other triangle in T?

02.9. (p. 20) Two magicians show the following trick. The first magician goes out of the room. The second magician takes a deck of 100 cards labelled by numbers $1, 2, \ldots, 100$ and asks three spectators to choose in turn one card each. The second magician sees what card each spectator has taken. Then he adds one more card from the rest of the deck. Spectators shuffle these four cards, call the first magician and give him these four cards. The first magician looks at the four cards and "guesses" what card was chosen by the first spectator, what card by the second and what card by the third. Prove that the magicians can perform this trick.

02.10. (p. 22) Let N be a positive integer. Two persons play the following game. The first player writes a list of positive integers not greater than 25, not necessarily different, such that their sum is at least 200. The second player wins if he can select some of these numbers so that their sum S satisfies the condition $200 - N \leq S \leq 200 + N$. What is the smallest value of N for which the second player has a winning strategy?

02.11. (p. 22) Let n be a positive integer. Consider n points in the plane such that no three of them are collinear and no two of the distances between them are equal. One by one, we connect each point to the two points nearest to it by line segments (if there are already other line segments drawn to this point, we do not erase these). Prove that there is no point from which line segments will be drawn to more than 11 points.

02.12. (p. 23) A set S of four distinct points is given in the plane. It is known that for any point $X \in S$ the remaining points can be denoted by Y, Z and W so that

$$XY = XZ + XW.$$

Prove that all the four points lie on a line.

02.13. (p. 23) Let ABC be an acute triangle with $\angle BAC > \angle BCA$, and let D be a point on the side AC such that $AB = BD$. Furthermore, let F be a point on the circumcircle of triangle ABC such that the line FD is perpendicular to the side BC and points F, B lie on different sides of the line AC. Prove that the line FB is perpendicular to the side AC.

02.14. (p. 24) Let L, M and N be points on sides AC, AB and BC of triangle ABC, respectively, such that BL is the bisector of angle ABC and segments AN, BL and CM have a common point. Prove that if $\angle ALB = \angle MNB$ then $\angle LNM = 90^\circ$.

02.15. (p. 24) A spider and a fly are sitting on a cube. The fly wants to maximize the shortest path to the spider along the surface of the cube. Is it necessarily best for the fly to be at the point opposite to the spider? ("Opposite" means "symmetric with respect to the centre of the cube".)

02.16. (p. 25) Find all nonnegative integers m such that

$$a_m = \left(2^{2m+1}\right)^2 + 1$$

is divisible by at most two different primes.

02.17. (p. 25) Show that the sequence

$$\binom{2002}{2002}, \binom{2003}{2002}, \binom{2004}{2002}, \ldots,$$

considered modulo 2002, is periodic.

02.18. (p. 26) Find all integers $n > 1$ such that any prime divisor of $n^6 - 1$ is a divisor of $(n^3 - 1)(n^2 - 1)$.

02.19. (p. 26) Let n be a positive integer. Prove that the equation

$$x + y + \frac{1}{x} + \frac{1}{y} = 3n$$

does not have solutions in positive rational numbers.

02.20. (p. 27) Does there exist an infinite non-constant arithmetic progression, each term of which is of the form a^b, where a and b are positive integers with $b \geq 2$?

Baltic Way 2003
Riga, Latvia
October 31–November 4, 2003

03.1. (p. 28) Let $\mathbb{Q}_+$ be the set of positive rational numbers. Find all functions $f\colon \mathbb{Q}_+ \to \mathbb{Q}_+$ which for all $x \in \mathbb{Q}_+$ fulfil

(1) $f(\frac{1}{x}) = f(x)$
(2) $(1 + \frac{1}{x})f(x) = f(x+1)$

03.2. (p. 28) Prove that any real solution of

$$x^3 + px + q = 0$$

satisfies the inequality $4qx \le p^2$.

03.3. (p. 28) Let x, y and z be positive real numbers such that $xyz = 1$. Prove that

$$(1+x)(1+y)(1+z) \ge 2\big(1 + \sqrt[3]{\frac{y}{x}} + \sqrt[3]{\frac{z}{y}} + \sqrt[3]{\frac{x}{z}}\big).$$

03.4. (p. 29) Let a, b, c be positive real numbers. Prove that

$$\frac{2a}{a^2+bc} + \frac{2b}{b^2+ca} + \frac{2c}{c^2+ab} \le \frac{a}{bc} + \frac{b}{ca} + \frac{c}{ab}.$$

03.5. (p. 29) A sequence (a_n) is defined as follows: $a_1 = \sqrt{2}$, $a_2 = 2$, and $a_{n+1} = a_n a_{n-1}^2$ for $n \ge 2$. Prove that for every $n \ge 1$ we have

$$(1+a_1)(1+a_2)\cdots(1+a_n) < (2+\sqrt{2})a_1 a_2 \cdots a_n.$$

03.6. (p. 30) Let $n \ge 2$ and $d \ge 1$ be integers with $d \mid n$, and let $x_1, x_2, \ldots, x_n$ be real numbers such that $x_1 + x_2 + \cdots + x_n = 0$. Prove that there are at least $\binom{n-1}{d-1}$ choices of d indices $1 \le i_1 < i_2 < \cdots < i_d \le n$ such that $x_{i_1} + x_{i_2} + \cdots + x_{i_d} \ge 0$.

03.7. (p. 30) Let X be a subset of $\{1, 2, 3, \ldots, 10000\}$ with the following property: If $a, b \in X$, $a \ne b$, then $a \cdot b \notin X$. What is the maximal number of elements in X?

03.8. (p. 31) There are 2003 pieces of candy on a table. Two players alternately make moves. A move consists of eating one candy or half of the candies on the table (the "lesser half" if there is an odd number of candies); at least one candy must be eaten at each move. The loser is the one who eats the last candy. Which player – the first or the second – has a winning strategy?

03.9. (p. 31) It is known that n is a positive integer, $n \le 144$. Ten questions of type "Is n smaller than a?" are allowed. Answers are given with a delay: The answer to the i'th question is given only after the $(i+1)$'st question is asked, $i = 1, 2, \ldots, 9$. The answer to the tenth question is given immediately after it is asked. Find a strategy for identifying n.

03.10. (p. 31) A *lattice point* in the plane is a point whose coordinates are both integral. The *centroid* of four points (x_i, y_i), $i = 1, 2, 3, 4$, is the point $(\frac{x_1+x_2+x_3+x_4}{4}, \frac{y_1+y_2+y_3+y_4}{4})$. Let n be the largest natural number with the following property: There are n distinct lattice points in the plane such that the centroid of any four of them is not a lattice point. Prove that $n = 12$.

03.11. (p. 32) Is it possible to select 1000 points in a plane so that at least 6000 distances between two of them are equal?

03.12. (p. 32) Let $ABCD$ be a square. Let M be an inner point on side BC and N be an inner point on side CD with $\angle MAN = 45^\circ$. Prove that the circumcentre of AMN lies on AC.

03.13. (p. 32) Let $ABCD$ be a rectangle and $BC = 2 \cdot AB$. Let E be the midpoint of BC and P an arbitrary inner point of AD. Let F and G be the feet of perpendiculars drawn correspondingly from A to BP and from D to CP. Prove that the points E, F, P, G are concyclic.

03.14. (p. 33) Let ABC be an arbitrary triangle and AMB, BNC, CKA regular triangles outward of ABC. Through the midpoint of MN a perpendicular to AC is constructed; similarly through the midpoints of NK resp. KM perpendiculars to AB resp. BC are constructed. Prove that these three perpendiculars intersect at the same point.

03.15. (p. 33) Let P be the intersection point of the diagonals AC and BD in a cyclic quadrilateral. A circle through P touches the side CD in the midpoint M of this side and intersects the segments BD and AC in the points Q and R, respectively. Let S be a point on the segment BD such that $BS = DQ$. The parallel to AB through S intersects AC at T. Prove that $AT = RC$.

03.16. (p. 34) Find all pairs of positive integers (a, b) such that $a - b$ is a prime and ab is a perfect square.

03.17. (p. 34) All the positive divisors of a positive integer n are stored into an array in increasing order. Mary has to write a program which decides for an arbitrarily chosen divisor $d > 1$ whether it is a prime. Let n have k divisors not greater than d. Mary claims that it suffices to check divisibility of d by the first $\lceil k/2 \rceil$ divisors of n: If a divisor of d greater than 1 is found among them, then d is composite, otherwise d is prime. Is Mary right?

03.18. (p. 35) Every integer is coloured with exactly one of the colours BLUE, GREEN, RED, YELLOW. Can this be done in such a way that if a, b, c, d are not all 0 and have the same colour, then $3a - 2b \neq 2c - 3d$?

03.19. (p. 35) Let a and b be positive integers. Prove that if $a^3 + b^3$ is the square of an integer, then $a + b$ is not a product of two different prime numbers.

03.20. (p. 36) Let n be a positive integer such that the sum of all the positive divisors of n (except n) plus the number of these divisors is equal to n. Prove that $n = 2m^2$ for some integer m.

Baltic Way 2004
Vilnius, Lithuania
November 5–November 9, 2004

04.1. (p. 37) Given a sequence $a_1, a_2, a_3, \dots$ of non-negative real numbers satisfying the conditions

(1) $a_n + a_{2n} \geq 3n$

(2) $a_{n+1} + n \leq 2\sqrt{a_n \cdot (n+1)}$

for all indices $n = 1, 2 \dots$.

(a) Prove that the inequality $a_n \geq n$ holds for every $n \in \mathbb{N}$.

(b) Give an example of such a sequence.

04.2. (p. 37) Let $P(x)$ be a polynomial with non-negative coefficients. Prove that if $P(\frac{1}{x})P(x) \geq 1$ for $x = 1$, then the same inequality holds for each positive x.

04.3. (p. 38) Let p, q, r be positive real numbers and $n \in \mathbb{N}$. Show that if $pqr = 1$, then

$$\frac{1}{p^n + q^n + 1} + \frac{1}{q^n + r^n + 1} + \frac{1}{r^n + p^n + 1} \leq 1.$$

04.4. (p. 38) Let $x_1, x_2, \dots, x_n$ be real numbers with arithmetic mean X. Prove that there is a positive integer K such that the arithmetic mean of each of the lists $\{x_1, x_2, \dots, x_K\}$, $\{x_2, x_3, \dots, x_K\}$, $\dots$, $\{x_{K-1}, x_K\}$, $\{x_K\}$ is not greater than X.

04.5. (p. 38) Determine the range of the function f defined for integers k by

$$f(k) = (k)_3 + (2k)_5 + (3k)_7 - 6k,$$

where $(k)_{2n+1}$ denotes the multiple of $2n + 1$ closest to k.

04.6. (p. 39) A positive integer is written on each of the six faces of a cube. For each vertex of the cube we compute the product of the numbers on the three adjacent faces. The sum of these products is 1001. What is the sum of the six numbers on the faces?

04.7. (p. 39) Find all sets X consisting of at least two positive integers such that for every pair $m, n \in X$, where $n > m$, there exists $k \in X$ such that $n = mk^2$.

04.8. (p. 39) Let f be a non-constant polynomial with integer coefficients. Prove that there is an integer n such that $f(n)$ has at least 2004 distinct prime factors.

04.9. (p. 40) A set S of $n - 1$ natural numbers is given ($n \geq 3$). There exists at least two elements in this set whose difference is not divisible by n. Prove that it is possible to choose a non-empty subset of S so that the sum of its elements is divisible by n.

04.10. (p. 40) Is there an infinite sequence of prime numbers $p_1, p_2, \dots$ such that $|p_{n+1} - 2p_n| = 1$ for each $n \in \mathbb{N}$?

04.11. (p. 40) An $m \times n$ table is given, in each cell of which a number $+1$ or -1 is written. It is known that initially exactly one -1 is in the table, all the other numbers being $+1$. During a move, it is allowed to choose any cell containing -1, replace this -1 by 0, and simultaneously multiply all the numbers in the neighboring cells by -1 (we say that two cells are neighboring if they have a common side). Find all (m, n) for which using such moves one can obtain the table containing zeroes only, regardless of the cell in which the initial -1 stands.

04.12. (p. 41) There are $2n$ different numbers in a row. By one move we can interchange any two numbers or interchange any three numbers cyclically (choose a, b, c and place a instead of b, b instead of c and c instead of a). What is the minimal number of moves that is always sufficient to arrange the numbers in increasing order?

04.13. (p. 41) The 25 member states of the European Union set up a committee with the following rules: (1) the committee should meet daily; (2) at each meeting, at least one member state should be represented; (3) at any two different meetings, a different set of member states should be represented; and (4) at the n'th meeting, for every $k < n$, the set of states represented should include at least one state that was represented at the k'th meeting. For how many days can the committee have its meetings?

04.14. (p. 41) We say that a pile is a set of four or more nuts. Two persons play the following game. They start with one pile of $n \geq 4$ nuts. During a move a player takes one of the piles that they have and split it into two non-empty subsets (these sets are not necessarily piles, they can contain an arbitrary number of nuts). If the player cannot move, he loses. For which values of n does the first player have a winning strategy?

04.15. (p. 42) A circle is divided into 13 segments, numbered consecutively from 1 to 13. Five fleas called A, B, C, D and E are sitting in the segments $1, 2, 3, 4$ and 5. A flea is allowed to jump to an empty segment five positions away in either direction around the circle. Only one flea jumps at the same time, and two fleas cannot be in the same segment. After some jumps, the fleas are back in the segments $1, 2, 3, 4, 5$, but possibly in some other order than they started. Which orders are possible?

04.16. (p. 43) Through a point P exterior to a given circle pass a secant and a tangent to the circle. The secant intersects the circle at A and B, and the tangent touches the circle at C on the same side of the diameter thorugh P as A and B. The projection of C on the diameter is Q. Prove that QC bisects $\angle AQB$.

04.17. (p. 43) Consider a rectangle with side lengths 3 and 4, and pick an arbitrary inner point on each side. Let x, y, z and u denote the side lengths of the quadrilateral spanned by these points. Prove that $25 \leq x^2 + y^2 + z^2 + u^2 \leq 50$.

04.18. (p. 43) A ray emanating from the vertex A of the triangle ABC intersects the side BC at X and the circumcircle of ABC at Y. Prove that $\frac{1}{AX} + \frac{1}{XY} \geq \frac{4}{BC}$.

04.19. (p. 43) D is the midpoint of the side BC of the given triangle ABC. M is a point on the side BC such that $\angle BAM = \angle DAC$. L is the second intersection point of the circumcircle of the triangle CAM with the side AB. K is the second intersection point of the circumcircle of the triangle BAM with the side AC. Prove that $KL \parallel BC$.

04.20. (p. 44) Three circular arcs w_1, w_2, w_3 with common endpoints A and B are on the same side of the line AB; w_2 lies between w_1 and w_3. Two rays emanating from B intersect these arcs at M_1, M_2, M_3 and K_1, K_2, K_3, respectively. Prove that $\frac{M_1M_2}{M_2M_3} = \frac{K_1K_2}{K_2K_3}$.

Baltic Way 2005
Stockholm, Sweden
November 3–November 7, 2005

05.1. (p. 45) Let a_0 be a positive integer. Define the sequence a_n, $n \geq 0$, as follows: If

$$a_n = \sum_{i=0}^{j} c_i 10^i$$

where c_i are integers with $0 \leq c_i \leq 9$, then

$$a_{n+1} = c_0^{2005} + c_1^{2005} + \cdots + c_j^{2005}.$$

Is it possible to choose a_0 so that all the terms in the sequence are distinct?

05.2. (p. 45) Let α, β and γ be three angles with $0 \leq \alpha, \beta, \gamma < 90^\circ$ and $\sin\alpha + \sin\beta + \sin\gamma = 1$. Show that

$$\tan^2\alpha + \tan^2\beta + \tan^2\gamma \geq \frac{3}{8}.$$

05.3. (p. 46) Consider the sequence a_k defined by $a_1 = 1$, $a_2 = \frac{1}{2}$,

$$a_{k+2} = a_k + \frac{1}{2}a_{k+1} + \frac{1}{4a_k a_{k+1}} \quad \text{for } k \geq 1.$$

Prove that

$$\frac{1}{a_1 a_3} + \frac{1}{a_2 a_4} + \frac{1}{a_3 a_5} + \cdots + \frac{1}{a_{98} a_{100}} < 4.$$

05.4. (p. 46) Find three different polynomials $P(x)$ with real coefficients such that $P(x^2 + 1) = P(x)^2 + 1$ for all real x.

05.5. (p. 47) Let a, b, c be positive real numbers with $abc = 1$. Prove that

$$\frac{a}{a^2+2} + \frac{b}{b^2+2} + \frac{c}{c^2+2} \leq 1.$$

05.6. (p. 47) Let K and N be positive integers with $1 \leq K \leq N$. A deck of N different playing cards is shuffled by repeating the operation of reversing the order of the K topmost cards and moving these to the bottom of the deck. Prove that the deck will be back in its initial order after a number of operations not greater than $4 \cdot N^2/K^2$.

05.7. (p. 48) A rectangular array has n rows and six columns, where $n > 2$. In each cell there is written either 0 or 1. All rows in the array are different from each other. For each pair of rows $(x_1, x_2, \ldots, x_6)$ and $(y_1, y_2, \ldots, y_6)$, the row $(x_1y_1, x_2y_2, \ldots, x_6y_6)$ can also be found in the array. Prove that there is a column in which at least half of the entries are zeroes.

05.8. (p. 48) Consider a grid of 25×25 unit squares. Draw with a red pen contours of squares of any size on the grid. What is the minimal number of squares we must draw in order to colour all the lines of the grid?

05.9. (p. 49) A rectangle is divided into 200×3 unit squares. Prove that the number of ways of splitting this rectangle into rectangles of size 1×2 is divisible by 3.

05.10. (p. 49) Let $m = 30030 = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13$ and let M be the set of its positive divisors which have exactly two prime factors. Determine the minimal integer n with the following property: for any choice of n numbers from M, there exist three numbers a, b, c among them satisfying $a \cdot b \cdot c = m$.

05.11. (p. 49) Let the points D and E lie on the sides BC and AC, respectively, of the triangle ABC, satisfying $BD = AE$. The line joining the circumcentres of the triangles ADC and BEC meets the lines AC and BC at K and L, respectively. Prove that $KC = LC$.

05.12. (p. 50) Let $ABCD$ be a convex quadrilateral such that $BC = AD$. Let M and N be the midpoints of AB and CD, respectively. The lines AD and BC meet the line MN at P and Q, respectively. Prove that $CQ = DP$.

05.13. (p. 50) What is the smallest number of circles of radius $\sqrt{2}$ that are needed to cover a rectangle

(a) of size 6×3?

(b) of size 5×3?

05.14. (p. 51) Let the medians of the triangle ABC meet at M. Let D and E be different points on the line BC such that $DC = CE = AB$, and let P and Q be points on the segments BD and BE, respectively, such that $2BP = PD$ and $2BQ = QE$. Determine $\angle PMQ$.

05.15. (p. 51) Let the lines e and f be perpendicular and intersect each other at O. Let A and B lie on e and C and D lie on f, such that all the five points A, B, C, D and O are distinct. Let the lines b and d pass through B and D respectively, perpendicularly to AC; let the lines a and c pass through A and C respectively, perpendicularly to BD. Let a and b intersect at X and c and d intersect at Y. Prove that XY passes through O.

05.16. (p. 52) Let p be a prime number and let n be a positive integer. Let q be a positive divisor of $(n+1)^p - n^p$. Show that $q - 1$ is divisible by p.

05.17. (p. 53) A sequence (x_n), $n \geq 0$, is defined as follows: $x_0 = a$, $x_1 = 2$ and $x_n = 2x_{n-1}x_{n-2} - x_{n-1} - x_{n-2} + 1$ for $n > 1$. Find all integers a such that $2x_{3n} - 1$ is a perfect square for all $n \geq 1$.

05.18. (p. 53) Let x and y be positive integers and assume that $z = 4xy/(x+y)$ is an odd integer. Prove that at least one divisor of z can be expressed in the form $4n - 1$ where n is a positive integer.

05.19. (p. 53) Is it possible to find 2005 different positive square numbers such that their sum is also a square number?

05.20. (p. 54) Find all positive integers $n = p_1 p_2 \cdots p_k$ which divide $(p_1+1)(p_2+1)\cdots(p_k+1)$, where $p_1 p_2 \cdots p_k$ is the factorization of n into prime factors (not necessarily distinct).

Baltic Way 2006
Turku, Finland
November 1–November 5, 2006

06.1. (p. 55) For a sequence $a_1, a_2, a_3, \ldots$ of real numbers it is known that
$$a_n = a_{n-1} + a_{n+2} \qquad \text{for } n = 2, 3, 4, \ldots.$$
What is the largest number of its consecutive elements that can all be positive?

06.2. (p. 55) Suppose that the real numbers $a_i \in [-2, 17]$, $i = 1, 2, \ldots, 59$, satisfy $a_1 + a_2 + \cdots + a_{59} = 0$. Prove that
$$a_1^2 + a_2^2 + \cdots + a_{59}^2 \leq 2006.$$

06.3. (p. 55) Prove that for every polynomial $P(x)$ with real coefficients there exist a positive integer m and polynomials $P_1(x), P_2(x), \ldots, P_m(x)$ with real coefficients such that
$$P(x) = \big(P_1(x)\big)^3 + \big(P_2(x)\big)^3 + \cdots + \big(P_m(x)\big)^3.$$

06.4. (p. 56) Let a, b, c, d, e, f be non-negative real numbers satisfying $a+b+c+d+e+f = 6$. Find the maximal possible value of
$$abc + bcd + cde + def + efa + fab$$
and determine all 6-tuples (a, b, c, d, e, f) for which this maximal value is achieved.

06.5. (p. 56) An occasionally unreliable professor has devoted his last book to a certain binary operation $*$. When this operation is applied to any two integers, the result is again an integer. The operation is known to satisfy the following axioms:

(a) $x * (x * y) = y$ for all $x, y \in \mathbb{Z}$;

(b) $(x * y) * y = x$ for all $x, y \in \mathbb{Z}$.

The professor claims in his book that

(C1) the operation $*$ is commutative: $x * y = y * x$ for all $x, y \in \mathbb{Z}$.

(C2) the operation $*$ is associative: $(x * y) * z = x * (y * z)$ for all $x, y, z \in \mathbb{Z}$.

Which of these claims follow from the stated axioms?

06.6. (p. 57) Determine the maximal size of a set of positive integers with the following properties:

(1) The integers consist of digits from the set $\{1, 2, 3, 4, 5, 6\}$.

(2) No digit occurs more than once in the same integer.

(3) The digits in each integer are in increasing order.

(4) Any two integers have at least one digit in common (possibly at different positions).

(5) There is no digit which appears in all the integers.

06.7. (p. 57) A photographer took some pictures at a party with 10 people. Each of the 45 possible pairs of people appears together on exactly one photo, and each photo depicts two or three people. What is the smallest possible number of photos taken?

06.8. (p. 57) The director has found out that six conspiracies have been set up in his department, each of them involving exactly three persons. Prove that the director can split the department in two laboratories so that none of the conspirative groups is entirely in the same laboratory.

06.9. (p. 58) To every vertex of a regular pentagon a real number is assigned. We may perform the following operation repeatedly: we choose two adjacent vertices of the pentagon and replace each of the two numbers assigned to these vertices by their arithmetic mean. Is it always possible to obtain the position in which all five numbers are zeroes, given that in the initial position the sum of all five numbers is equal to zero?

06.10. (p. 58) 162 pluses and 144 minuses are placed in a 30×30 table in such a way that each row and each column contains at most 17 signs. (No cell contains more than one sign.) For every plus we count the number of minuses in its row and for every minus we count the number of pluses in its column. Find the maximum of the sum of these numbers.

06.11. (p. 58) The altitudes of a triangle are 12, 15 and 20. What is the area of the triangle?

06.12. (p. 59) Let ABC be a triangle, let B_1 be the midpoint of the side AB and C_1 the midpoint of the side AC. Let P be the point of intersection, other than A, of the circumscribed circles around the triangles ABC_1 and AB_1C. Let P_1 be the point of intersection, other than A, of the line AP with the circumscribed circle around the triangle AB_1C_1. Prove that $2AP = 3AP_1$.

06.13. (p. 59) In a triangle ABC, points D, E lie on sides AB, AC respectively. The lines BE and CD intersect at F. Prove that if

$$BC^2 = BD \cdot BA + CE \cdot CA,$$

then the points A, D, F, E lie on a circle.

06.14. (p. 59) There are 2006 points marked on the surface of a sphere. Prove that the surface can be cut into 2006 congruent pieces so that each piece contains exactly one of these points inside it.

06.15. (p. 60) Let the medians of the triangle ABC intersect at the point M. A line t through M intersects the circumcircle of ABC at X and Y so that A and C lie on the same side of t. Prove that $BX \cdot BY = AX \cdot AY + CX \cdot CY$.

06.16. (p. 61) Are there four distinct positive integers such that adding the product of any two of them to 2006 yields a perfect square?

06.17. (p. 61) Determine all positive integers n such that $3^n + 1$ is divisible by n^2.

06.18. (p. 61) For a positive integer n let a_n denote the last digit of $n^{(n^n)}$. Prove that the sequence (a_n) is periodic and determine the length of the minimal period.

06.19. (p. 61) Does there exist a sequence $a_1, a_2, a_3, \dots$ of positive integers such that the sum of every n consecutive elements is divisible by n^2 for every positive integer n?

06.20. (p. 62) A 12-digit positive integer consisting only of digits 1, 5 and 9 is divisible by 37. Prove that the sum of its digits is not equal to 76.

SOLUTIONS

Baltic Way 2002

02.1. *Solve the system of equations*

$$\begin{aligned} a^3 + 3ab^2 + 3ac^2 - 6abc = 1 \\ b^3 + 3ba^2 + 3bc^2 - 6abc = 1 \\ c^3 + 3ca^2 + 3cb^2 - 6abc = 1 \end{aligned}$$

in real numbers.
Answer: $a = 1$, $b = 1$, $c = 1$.
Solution: Denoting the left-hand sides of the given equations as A, B and C, the following equalities can easily be seen to hold:

$$\begin{aligned} -A + B + C &= (-a + b + c)^3 \\ A - B + C &= (a - b + c)^3 \\ A + B - C &= (a + b - c)^3 \end{aligned}$$

Hence, the system of equations given in the problem is equivalent to

$$(-a + b + c)^3 = 1, \qquad (a - b + c)^3 = 1, \qquad (a + b - c)^3 = 1,$$

which gives

$$-a + b + c = 1, \qquad a - b + c = 1, \qquad a + b - c = 1.$$

The unique solution of this system is $(a, b, c) = (1, 1, 1)$.
02.2. *Let a, b, c, d be real numbers such that*

$$a + b + c + d = -2, \tag{02.1}$$
$$ab + ac + ad + bc + bd + cd = 0. \tag{02.2}$$

Prove that at least one of the numbers a, b, c, d is not greater than -1.
Solution: We can assume that a is the least among a, b, c, d (or one of the least, if some of them are equal), there are $n > 0$ negative numbers among a, b, c, d, and the sum of the positive ones is x.

Then we obtain

$$-2 = a + b + c + d \geq na + x. \tag{02.3}$$

Squaring we get $4 = a^2 + b^2 + c^2 + d^2$, which implies

$$4 \leq n \cdot a^2 + x^2 \tag{02.4}$$

as the square of the sum of positive numbers is not less than the sum of their squares.

The inequality (02.3) implies that $-x \geq na + 2$, and since both sides are negative we get $x^2 \leq (na + 2)^2$. Combining this with (02.4) we obtain $na^2 + (na + 2)^2 \geq 4$, which implies that

$$a^2 + na^2 + 4a \geq 0.$$

As $n \leq 3$ (if all the numbers are negative, the second condition of the problem cannot be satisfied), we obtain from the last inequality that $4a^2 + 4a \geq 0$, whence

$$a(a+1) \geq 0.$$

As $a < 0$ it follows that $a \leq -1$.

Solution 2: Assume that $a, b, c, d > -1$. Denoting $A = a+1$, $B = b+1$, $C = c+1$, $D = d+1$ we have $A, B, C, D > 0$. Then the first equation gives

$$A + B + C + D = 2. \tag{02.5}$$

We also have

$$ab = (A-1)(B-1) = AB - A - B + 1.$$

Adding five similar terms to the last one we get from the second equation

$$AB + AC + AD + BC + BD + CD - 3(A + B + C + D) + 6 = 0.$$

In view of (02.5) this implies

$$AB + AC + AD + BC + BD + CD = 0,$$

a contradiction as all the unknowns A, B, C, D were supposed to be positive.

Solution 3: Assume that the conditions (02.1) and (02.2) of the problem hold, and that

$$a, b, c, d > -1. \tag{02.6}$$

If all of a, b, c, d were negative, then (02.2) could not be satisfied, so at most three of them are negative. If two or less of them were negative, then (02.6) would imply that the sum of negative numbers, and hence also the sum $a + b + c + d$, is greater than $2 \cdot (-1) = -2$, which contradicts (02.1). So exactly three of a, b, c, d are negative and one is nonnegative. Let d be the nonnegative one. Then $d = -2 - (a + b + c) < -2 - (-1 - 1 - 1) = 1$. Obviously $|a|, |b|, |c|, |d| < 1$. Squaring (02.1) and subtracting 2 times (02.2), we get

$$a^2 + b^2 + c^2 + d^2 = 4,$$

but

$$a^2 + b^2 + c^2 + d^2 = |a|^2 + |b|^2 + |c|^2 + |d|^2 < 4,$$

a contradiction.

02.3. *Find all sequences $a_0 \leq a_1 \leq a_2 \leq \cdots$ of real numbers such that*

$$a_{m^2+n^2} = a_m^2 + a_n^2 \tag{02.7}$$

for all integers $m, n \geq 0$.

Answer: $a_n \equiv 0$, $a_n \equiv \frac{1}{2}$ and $a_n = n$.

Solution: Substituting $m = n = 0$ into (02.7) we get $a_0 = 2a_0^2$, hence either $a_0 = \frac{1}{2}$ or $a_0 = 0$. We consider these cases separately.

(1) If $a_0 = \frac{1}{2}$ then substituting $m = 1$ and $n = 0$ into (02.7) we obtain $a_1 = a_1^2 + \frac{1}{4}$, whence $\left(a_1 - \frac{1}{2}\right)^2 = 0$ and $a_1 = \frac{1}{2}$. Now,

$$a_2 = a_{1^2+1^2} = 2a_1^2 = \tfrac{1}{2},$$
$$a_8 = a_{2^2+2^2} = 2a_2^2 = \tfrac{1}{2},$$

etc., implying that $a_{2^i} = \frac{1}{2}$ for arbitrarily large natural i and, due to monotonicity, $a_n = \frac{1}{2}$ for every natural n.

(2) If $a_0 = 0$ then by substituting $m = 1$, $n = 0$ into (02.7) we obtain $a_1 = a_1^2$ and hence, $a_1 = 0$ or $a_1 = 1$. This gives two subcases.

(2a) If $a_0 = 0$ and $a_1 = 0$ then by the same technique as above we see that $a_{2^i} = 0$ for arbitrarily large natural i and, due to monotonicity, $a_n = 0$ for every natural n.

(2b) If $a_0 = 0$ and $a_1 = 1$ then we compute

$$\begin{aligned} a_2 &= a_{1^2+1^2} = 2a_1^2 = 2, \\ a_4 &= a_{2^2+0^2} = a_2^2 = 4, \\ a_5 &= a_{2^2+1^2} = a_2^2 + a_1^2 = 5. \end{aligned}$$

Now,

$$a_3^2 + a_4^2 = a_{25} = a_5^2 + a_0^2 = 25,$$

hence $a_3^2 = 25 - 16 = 9$ and $a_3 = 3$. Further,

$$\begin{aligned} a_8 &= a_{2^2+2^2} = 2a_2^2 = 8, \\ a_9 &= a_{3^2+0^2} = a_3^2 = 9, \\ a_{10} &= a_{3^2+1^2} = a_3^2 + a_1^2 = 10. \end{aligned}$$

From the equalities

$$\begin{aligned} a_6^2 + a_8^2 &= a_{10}^2 + a_0^2, \\ a_7^2 + a_1^2 &= a_5^2 + a_5^2, \end{aligned}$$

we also conclude that $a_6 = 6$ and $a_7 = 7$. It remains to note that

$$\begin{aligned} (2k+1)^2 + (k-2)^2 &= (2k-1)^2 + (k+2)^2, \\ (2k+2)^2 + (k-4)^2 &= (2k-2)^2 + (k+4)^2, \end{aligned}$$

and by induction it follows that $a_n = n$ for every natural n.

02.4. *Let n be a positive integer. Prove that*

$$\sum_{i=1}^{n} x_i(1-x_i)^2 \le \left(1 - \tfrac{1}{n}\right)^2$$

for all nonnegative real numbers $x_1, x_2, \ldots, x_n$ such that $x_1 + x_2 + \cdots + x_n = 1$.

Solution: Expanding the expressions at both sides we obtain the equivalent inequality

$$-\sum_i x_i^3 + 2\sum_i x_i^2 - \frac{2}{n} + \frac{1}{n^2} \ge 0.$$

It is easy to check that the left-hand side is equal to

$$\sum_i \left(2 - \tfrac{2}{n} - x_i\right)\left(x_i - \tfrac{1}{n}\right)^2$$

and hence is nonnegative.

Solution 2: First note that for $n = 1$ the required condition holds trivially, and for $n = 2$ we have

$$x(1-x)^2 + (1-x)x^2 = x(1-x) \leq \Big(\frac{x+(1-x)}{2}\Big)^2 = \Big(1-\frac{1}{2}\Big)^2.$$

We now consider the case $n \geq 3$.

Assume first that for each index i the inequality $x_i < \frac{2}{3}$ holds. Let $f(x) = x(1-x)^2 = x - 2x^2 + x^3$, then $f''(x) = 6x - 4$. Hence, the function f is concave in the interval $\big[0, \frac{2}{3}\big]$. Thus, from Jensen's inequality we have

$$\begin{aligned}\sum_{i=1}^{n} x_i(1-x_i)^2 = \sum_{i=1}^{n} f(x_i) &\leq n \cdot f\Big(\frac{x_1+\cdots+x_n}{n}\Big) = n \cdot f(\tfrac{1}{n}) \\ &= n \cdot \tfrac{1}{n}\big(1-\tfrac{1}{n}\big)^2 = \big(1-\tfrac{1}{n}\big)^2.\end{aligned}$$

If some $x_i \geq \frac{2}{3}$ then we have

$$x_i(1-x_i)^2 \leq 1 \cdot (1-\tfrac{2}{3})^2 = \tfrac{1}{9}.$$

For the rest of the terms we have

$$\sum_{j \neq i} x_j(1-x_j)^2 \leq \sum_{j \neq i} x_j = 1 - x_i \leq \tfrac{1}{3}.$$

Hence,

$$\sum_{i=1}^{n} x_i(1-x_i)^2 \leq \tfrac{1}{9} + \tfrac{1}{3} = \tfrac{4}{9} \leq \big(1-\tfrac{1}{n}\big)^2$$

as $n \geq 3$.

02.5. *Find all pairs (a, b) of positive rational numbers such that*

$$\sqrt{a} + \sqrt{b} = \sqrt{2+\sqrt{3}}.$$

Answer: The two solutions are $(a, b) = \big(\frac{1}{2}, \frac{3}{2}\big)$ and $(a, b) = \big(\frac{3}{2}, \frac{1}{2}\big)$.

Solution: Squaring both sides of the equation gives

$$a + b + 2\sqrt{ab} = 2 + \sqrt{3} \qquad (02.8)$$

so $2\sqrt{ab} = r + \sqrt{3}$ for some rational number r. Squaring both sides of this gives $4ab = r^2 + 3 + 2r\sqrt{3}$, so $2r\sqrt{3}$ is rational, which implies $r = 0$. Hence $ab = 3/4$ and substituting this into (02.8) gives $a + b = 2$. Solving for a and b gives $(a, b) = \big(\frac{1}{2}, \frac{3}{2}\big)$ or $(a, b) = \big(\frac{3}{2}, \frac{1}{2}\big)$.

02.6. *The following solitaire game is played on an $m \times n$ rectangular board, $m, n \geq 2$, divided into unit squares. First, a rook is placed on some square. At each move, the rook can be moved an arbitrary number of squares horizontally or vertically, with the extra condition that each move has to be made in the 90° clockwise direction compared to the previous one (e.g. after a move to the left, the next one has to be done upwards, the next one to the right etc.). For which values of m and n is it possible that the rook visits every square of the board exactly once and returns to the first square? (The rook is considered to visit only those squares it stops on, and not the ones it steps over.)*

Answer: It is possible precisely when m and n are even.

Solution: First, consider any row that is not the row where the rook starts from. The rook has to visit all the squares of that row exactly once, and on its tour around the board, every time it visits this row, exactly two squares get visited. Hence, m must be even; a similar argument for the columns shows that n must also be even.

It remains to prove that for any even m and n such a tour is possible. We will show it by an induction-like argument. Labelling the squares with pairs of integers (i, j), where $1 \le i \le m$ and $1 \le j \le n$, we start moving from the square $(1, \frac{n}{2} + 1)$ and first cover all the squares of the first and last columns in the order shown in the figure to the right, except for the squares $(m, \frac{n}{2})$ and $(m, \frac{n}{2} + 1)$; note that we finish on the square $(1, \frac{n}{2})$.

2							3
6							7
10							
1							
9							8
5							4

2	14	22	34	35	23	15	3
6	18	26	38	39	27	19	7
10		30		31		**11**	
1		21		**40**		20	
9	17	29	37	36	28	16	8
5	13	25	33	32	24	12	4

The next square to visit is $(m - 1, \frac{n}{2})$ and now we will cover the columns numbered 2 and $m - 1$, except for the two middle squares in column 2. Continuing in this way we can visit all the squares except for the two middle squares in every second column (note that here we need the assumption that m and n are even). The rest of the squares can be visited easily.

02.7. *We draw n convex quadrilaterals in the plane. They divide the plane into regions (one of the regions is infinite). Determine the maximal possible number of these regions.*

Answer: The maximal number of regions is $4n^2 - 4n + 2$.

Solution: One quadrilateral produces two regions. Suppose we have drawn k quadrilaterals $Q_1, \ldots, Q_k$ and produced a_k regions. We draw another quadrilateral Q_{k+1} and try to evaluate the number of regions a_{k+1} now produced. Note that if a vertex of Q_i is located on an edge of another quadrilateral then it is possible to move this vertex a little bit to obtain a configuration with the same number of regions or one more region. Hence there exists a maximal configuration where no vertex of any Q_i is located on an edge of another quadrilateral, and in the following, looking only for the maximal number of regions, we can assume that no vertex of a quadrilateral is located on an edge of another quadrilateral.

Because of this fact and the convexity of the Q_j's, any one of the four sides of Q_{k+1} meets at most two sides of any Q_j. So the sides of Q_{k+1} are divided into at most $2k + 1$ segments, each of which potentially increases the number of regions by one (being part of the common boundary of two parts, one of which is counted in a_k).

But if a side of Q_{k+1} intersects the boundary of each Q_j, $1 \le j \le k$ twice, then its endpoints (vertices of Q_{k+1}) are in the region outside of all the Q_j's, and the segments meeting at such a vertex are on the boundary of a single new part (recall that it makes no sense to put vertices on edges of another quadrilaterals). This means that $a_{k+1} - a_k \le 4(2k + 1) - 4 = 8k$. By considering squares inscribed in a circle one easily sees that the situation where $a_{k+1} - a_k = 8k$ can be reached.

It remains to determine the expression for the maximal a_k. Since the difference $a_{k+1} - a_k$ is linear in k, a_k is a quadratic polynomial in k, and $a_0 = 2$. So $a_k = Ak^2 + Bk + 2$. We have $8k = a_{k+1} - a_k = A(2k + 1) + B$ for all k. This implies $A = 4$, $B = -4$, and $a_n = 4n^2 - 4n + 2$.

02.8. *Let P be a set of $n \ge 3$ points in the plane, no three of which are on a line. How many possibilities are there to choose a set T of $\binom{n-1}{2}$ triangles, whose vertices are all in P, such that each triangle in T has a side that is not a side of any other triangle in T?*

Answer: There is one possibility for $n = 3$ and n possibilities for $n \ge 4$.

Solution: For a fixed point $x \in P$, let T_x be the set of all triangles with vertices in P which have x as a vertex. Clearly, $|T_x| = \binom{n-1}{2}$, and each triangle in T_x has a side which is not a side of any other triangle in T_x. For any $x, y \in P$ such that $x \ne y$, we have $T_x \ne T_y$ if and only if $n \ge 4$. We will show that any possible set T is equal to T_x for some $x \in P$, that is, that the answer is 1 for $n = 3$ and n for $n \ge 4$.

Let

$$T = \left\{ t_i \mid i = 1, 2, \ldots, \binom{n-1}{2} \right\} \qquad\qquad S = \left\{ s_i \mid i = 1, 2, \ldots, \binom{n-1}{2} \right\}$$

such that T is a set of triangles whose vertices are all in P, and s_i is a side of t_i but not of any t_j, $j \neq i$. Furthermore, let C be the collection of all the $\binom{n}{3}$ triangles whose vertices are in P. Note that

$$|C \setminus T| = \binom{n}{3} - \binom{n-1}{2} = \binom{n-1}{3}.$$

Let m be the number of pairs (s, t) such that $s \in S$ is a side of $t \in C \setminus T$. Since every $s \in S$ is a side of exactly $n - 3$ triangles from $C \setminus T$, we have

$$m = |S| \cdot (n-3) = \binom{n-1}{2} \cdot (n-3) = 3 \cdot \binom{n-1}{3} = 3 \cdot |C \setminus T|.$$

On the other hand, every $t \in C \setminus T$ has at most three sides from S. By the above equality, for every $t \in C \setminus T$, all its sides must be in S.

Assume that for $p \in P$ there is a side $s \in S$ such that p is an endpoint of s. Then p is also a vertex of each of the $n - 3$ triangles in $C \setminus T$ which have s as a side. Consequently, p is an endpoint of $n - 2$ sides in S. Since every side in S has exactly 2 endpoints, the number of points $p \in P$ which occur as a vertex of some $s \in S$ is

$$\frac{2 \cdot |S|}{n-2} = \frac{2}{n-2} \cdot \binom{n-1}{2} = n - 1.$$

Consequently, there is an $x \in P$ which is not an endpoint of any $s \in S$, and hence T must be equal to T_x.

02.9. *Two magicians show the following trick. The first magician goes out of the room. The second magician takes a deck of* 100 *cards labelled by numbers* $1, 2, \ldots, 100$ *and asks three spectators to choose in turn one card each. The second magician sees what card each spectator has taken. Then he adds one more card from the rest of the deck. Spectators shuffle these four cards, call the first magician and give him these four cards. The first magician looks at the four cards and "guesses" what card was chosen by the first spectator, what card by the second and what card by the third. Prove that the magicians can perform this trick.*

Solution: We will identify ourselves with the second magician. Then we need to choose a card in such a manner that another magician will be able to understand which of the four cards we have chosen and what information it gives about the order of the other cards. We will reach these two goals independently.

Let a, b, c be the remainders of the labels of the spectators' three cards modulo 5. There are three possible cases.

(1) *All the three remainders coincide.* Then choose a card with a remainder not equal to the remainder of the spectators' cards. Denote this remainder d.

Note that we now have *two different remainders, one of them in three copies* (this will be used by the first magician to distinguish between the three cases). To determine which of the cards is chosen by us is now a simple exercise in division by 5. But we must also encode the ordering of the spectators' cards. These cards have a natural ordering by their labels, and they are also ordered by their belonging to the spectators. Thus, we have to encode a permutation of three elements. There are six permutations of three elements, let us enumerate them somehow. Then, if we want to inform the first magician that spectators form a permutation number k with respect to the natural ordering, we choose the card number $5k + d$.

(2) *The remainders a, b, c are pairwise different.* Then it is clear that, modulo 5, exactly one of the following possibilities takes place:

$$|b-a| \equiv |a-c|, \quad |a-b| \equiv |b-c|, \quad \text{or} \quad |a-c| \equiv |c-b| \tag{02.9}$$

It is not hard to prove this by a case study, but one could also imagine choosing three vertices of a regular pentagon – these vertices always form an isosceles, but not an equilateral triangle.

Each of these possibilities has one of the remainders distinguished from the other two remainders (these distinguished remainders are a, b, c, respectively). Now, choose a card from the rest of the deck having the distinguished remainder modulo 5. Hence, we have *three different remainders, one of them distinguished by* (02.9) *and presented in two copies.* Let d be the distinguished remainder and $s = 5m + d$ be the spectator's card with this remainder.

Now we have to choose a card r with the remainder d such that the first magician would be able to understand which of the cards s and r was chosen by us and what permutation of the spectators it implies. This can be done easily: If we want to inform the first magician that the spectators form a permutation number k with respect to the natural ordering, we choose the card number $s + 5k \pmod{100}$.

The decoding procedure is easy: If we have two numbers p and q that have the same remainder modulo 5, calculate $p - q \mod 100$ and $q - p \mod 100$. If $p - q \mod 100 > q - p \mod 100$ then $r = q$ is our card and $s = p$ is the spectator's card. (The case $p - q \mod 100 = q - p \mod 100$ is impossible since the sum of these numbers is equal to 100, and one of them is not greater than $6 \cdot 5 = 30$.)

(3) *Two remainders (say, a and b) coincide.* Let us choose a card with the remainder $d = (a + c)/2 \mod 5$. Then $|a - d| = |d - c| \mod 5$, so the remainder d is distinguished by (02.9). Hence we have *three different remainders, one of them distinguished by* (02.9) *and one of the non-distinguished remainders presented in two copies.* The first magician will easily determine our card, and the rule to choose the card in order to enable him also determine the order of spectators is similar to the one in the first case.

Solution 2: This solution gives a non-constructive proof that the trick is possible. For this, we need to show there is an injective mapping from the set of ordered triples to the set of unordered quadruples that additionally respects inclusion.

To prove that the desired mapping exists, let's consider a bipartite graph such that the set of ordered triples T and the set of unordered quadruples Q form the two disjoint sets of vertices and there is an edge between a triple and a quadruple if and only if the triple is a subset of the quadruple.

For each triple $t \in T$, we can add any of the remaining 97 cards to it, and thus we have 97 different quadruples connected to each triple in the graph. Conversely, for each quadruple $q \in Q$, we can remove any of the four cards from it, and reorder the remaining three cards in $3! = 6$ different ways, and thus we have 24 different triples connected to each quadruple in the graph.

According to the Hall's theorem, a bipartite graph (T, Q, E) has a perfect matching if and only if for each subset $T' \subseteq T$ the set of neighbours of T', denoted $N(T')$, satisfies $|N(T')| \geq |T'|$.

To prove that this condition holds for our graph, consider any subset $T' \subseteq T$. Because we have 97 quadruples for each triple, and there can be at most 24 copies of each of them in the multiset of neighbours, we have $|N(T')| \geq \frac{97}{24}|T'| > 4|T'|$, which is even much more than we need. Thus, the desired mapping is guaranteed to exist.

Solution 3: Let the three chosen numbers be (x_1, x_2, x_3). At least one of the sets $\{1, 2, \ldots, 24\}$, $\{25, 26, \ldots, 48\}$, $\{49, 50, \ldots, 72\}$ and $\{73, 74, \ldots, 96\}$ contains none of x_1, x_2, x_3; let S be such set. We split S into six parts $S = S_1 \cup S_2 \cup \cdots \cup S_6$ so that the first four elements of S are in S_1, the four next in S_2, etc. Now we choose $i \in \{1, 2, \ldots, 6\}$ corresponding to the lexicographic order of the numbers x_1, x_2, x_3 (if $x_1 < x_2 < x_3$ then $i = 1$, if $x_1 < x_3 < x_2$ then $i = 2$, …, if $x_3 < x_2 < x_1$ then $i = 6$). At last let j be the number of elements in $\{x_1, x_2, x_3\}$ that are greater than elements of S (note that any x_k, $k = 1, 2, 3$, is either greater or smaller than all the elements of S). Now we choose $x_4 \in S_i$ so that $x_1 + x_2 + x_3 + x_4 \equiv j \pmod 4$ and add the card number x_4 to those three cards.

Decoding of $\{a, b, c, d\}$ is straightforward. We first put the numbers into increasing order and then calculate $a + b + c + d \mod 4$ showing the added card. The added card belongs to some S_i ($i \in \{1, 2, \ldots, 6\}$) for some S and i shows us the initial ordering of cards.

02.10. *Let N be a positive integer. Two persons play the following game. The first player writes a list of positive integers not greater than 25, not necessarily different, such that their sum is at least 200. The second player wins if he can select some of these numbers so that their sum S satisfies the condition $200 - N \leq S \leq 200 + N$. What is the smallest value of N for which the second player has a winning strategy?*

Answer: $N = 11$.

Solution: If $N = 11$, then the second player can simply remove numbers from the list, starting with the smallest number, until the sum of the remaining numbers is less than 212. If the last number removed was not 24 or 25, then the sum of the remaining numbers is at least $212 - 23 = 189$. If the last number removed was 24 or 25, then only 24's and 25's remain, and there must be exactly 8 of them since their sum must be less than 212 and not less than $212 - 24 = 188$. Hence their sum S satisfies $8 \cdot 24 = 192 \leq S \leq 8 \cdot 25 = 200$. In any case the second player wins.

On the other hand, if $N \leq 10$, then the first player can write 25 two times and 23 seven times. Then the sum of all numbers is 211, but if at least one number is removed, then the sum of the remaining ones is at most 188 – so the second player cannot win.

02.11. *Let n be a positive integer. Consider n points in the plane such that no three of them are collinear and no two of the distances between them are equal. One by one, we connect each point to the two points nearest to it by line segments (if there are already other line segments drawn to this point, we do not erase these). Prove that there is no point from which line segments will be drawn to more than 11 points.*

Solution: Suppose there exists a point A such that A is connected to 12 points. Then there exist three points B, C and D such that $\angle BAC \leq 60^\circ$, $\angle BAD \leq 60^\circ$ and $\angle CAD \leq 60^\circ$.

We can assume that $AD > AB$ and $AD > AC$. By the cosine law we have

$$\begin{aligned} BD^2 &= AD^2 + AB^2 - 2AD AB \cos \angle BAD \\ &< AD^2 + AB^2 - 2AB^2 \cos \angle BAD \\ &= AD^2 + AB^2(1 - 2\cos \angle BAD) \\ &\leq AD^2 \end{aligned}$$

since $1 \leq 2\cos(\angle BAD)$. Hence $BD < AD$. Similarly we get $CD < AD$. Hence A and D should not be connected which is a contradiction.

Remark: It would be interesting to know whether 11 can be achieved or the actual bound is lower.

02.12. *A set S of four distinct points is given in the plane. It is known that for any point $X \in S$ the remaining points can be denoted by Y, Z and W so that*

$$XY = XZ + XW.$$

Prove that all the four points lie on a line.
Solution: Let $S = \{A, B, C, D\}$ and let AB be the longest of the six segments formed by these four points (if there are several longest segments, choose any of them). If we choose $X = A$ then we must also choose $Y = B$. Indeed, if we would, for example, choose $Y = C$, we should have $AC = AB + AD$ contradicting the maximality of AB. Hence we get

$$AB = AC + AD. \tag{02.10}$$

Similarly, choosing $X = B$ we must choose $Y = A$ and we obtain

$$AB = BC + BD. \tag{02.11}$$

On the other hand, from the triangle inequality we know that

$$AB \leq AC + BC,$$
$$AB \leq AD + BD,$$

where at least one of the inequalities is strict if all the four points are not on the same line. Hence, adding the two last inequalities we get

$$2AB < AC + BC + AD + BD.$$

On the other hand, adding (02.10) and (02.11) we get

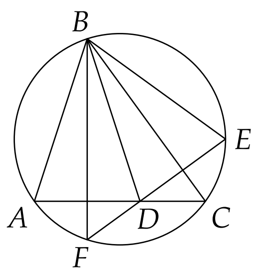
$$2AB = AC + AD + BC + BD;$$

a contradiction.

02.13. *Let ABC be an acute triangle with $\angle BAC > \angle BCA$, and let D be a point on the side AC such that $AB = BD$. Furthermore, let F be a point on the circumcircle of triangle ABC such that the line FD is perpendicular to the side BC and points F, B lie on different sides of the line AC. Prove that the line FB is perpendicular to the side AC.*
Solution: Let E be the other point on the circumcircle of triangle ABC such that $AB = EB$. Let D' be the point of intersection of side AC and the line perpendicular to side BC, passing through E. Then $\angle ECB = \angle BCA$ and the triangle ECD' is isosceles. As $ED' \perp BC$, the triangle BED' is also isosceles and $BE = BD'$ implying $D = D'$. Hence, the points E, D, F lie on one line. We now have

$$\angle EFB + \angle FDA = \angle BCA + \angle EDC = 90^\circ.$$

The required result now follows.



02.14. *Let L, M and N be points on sides AC, AB and BC of triangle ABC, respectively, such that BL is the bisector of angle ABC and segments AN, BL and CM have a common point. Prove that if $\angle ALB = \angle MNB$ then $\angle LNM = 90°$.*

Solution: Let P be the intersection point of lines MN and AC. Then $\angle PLB = \angle PNB$ and the quadrangle $PLNB$ is cyclic. Let ω be its circumcircle. It is sufficient to prove that PL is a diameter of ω.

Let Q denote the second intersection point of the line AB and ω. Then $\angle PQB = \angle PLB$ and

$$\angle QPL = \angle QBL = \angle LBN = \angle LPN,$$

and the triangles PAQ and BAL are similar. Therefore,

$$\frac{PQ}{PA} = \frac{BL}{BA}. \tag{02.12}$$

We see that the line PL is a bisector of the inscribed angle NPQ. Now in order to prove that PL is a diameter of ω it is sufficient to check that $PN = PQ$.

The triangles NPC and LBC are similar, hence

$$\frac{PN}{PC} = \frac{BL}{BC}. \tag{02.13}$$

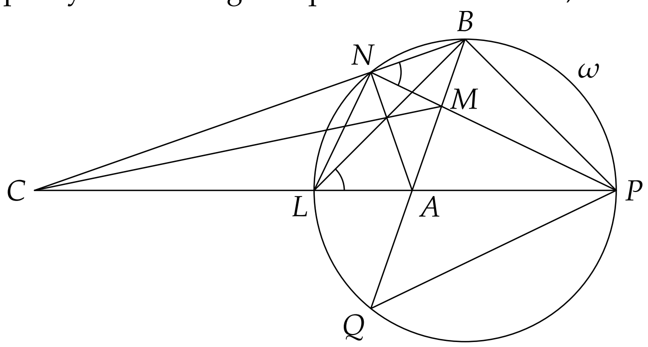
Note also that

$$\frac{AB}{BC} = \frac{AL}{CL}. \tag{02.14}$$

by the properties of a bisector. Combining (02.12), (02.13) and (02.14) we have

$$\frac{PN}{PQ} = \frac{AL}{AP} \cdot \frac{CP}{CL}.$$

We want to prove that the left-hand side of this equality equals 1. This follows from the fact that the quadruple of points (C, A, L, P) is harmonic, as can be proven using standard methods (for example by considering the quadrilateral $MBNS$, where $S = MC \cap AN$).



02.15. *A spider and a fly are sitting on a cube. The fly wants to maximize the shortest path to the spider along the surface of the cube. Is it necessarily best for the fly to be at the point opposite to the spider? ("Opposite" means "symmetric with respect to the centre of the cube".)*

Answer: No.

Solution: Suppose that the side of the cube is 1 and the spider sits at the middle of one of the edges. Then the shortest path to the middle of the opposite edge has length 2.

However, if the fly goes to a point on this edge at distance s from the middle, then the length of the shortest path is

$$\min\Big(\sqrt{4+s^2}, \sqrt{\tfrac{9}{4}+\big(\tfrac{3}{2}-s\big)^2}\Big).$$

If $0 < s < (3-\sqrt{7})/2$, then this expression is greater than 2.

02.16. *Find all nonnegative integers m such that*

$$a_m = \left(2^{2m+1}\right)^2 + 1$$

is divisible by at most two different primes.

Solution: Obviously $m = 0, 1, 2$ are solutions as $a_0 = 5$, $a_1 = 65 = 5 \cdot 13$, and $a_2 = 1025 = 25 \cdot 41$. We show that these are the only solutions.

Assume that $m \geq 3$ and that a_m contains at most two different prime factors. Clearly, $a_m = 4^{2m+1} + 1$ is divisible by 5, and

$$a_m = \big(2^{2m+1} + 2^{m+1} + 1\big) \cdot \big(2^{2m+1} - 2^{m+1} + 1\big).$$

The two above factors are relatively prime as they are both odd and their difference is a power of 2. Since both factors are larger than 1, one of them must be a power of 5. Hence,

$$2^{m+1} \cdot \big(2^m \pm 1\big) = 5^t - 1 = (5-1) \cdot (1 + 5 + \cdots + 5^{t-1})$$

for some positive integer t, where $\pm$ reads as *either plus or minus*. For odd t the right-hand side is not divisible by 8, contradicting $m \geq 3$. Therefore, t must be even and

$$2^{m+1} \cdot \big(2^m \pm 1\big) = (5^{t/2} - 1) \cdot (5^{t/2} + 1).$$

Clearly, $5^{t/2} + 1 \equiv 2 \pmod 4$. Consequently, $5^{t/2} - 1 = 2^m \cdot k$ for some odd k, and $5^{t/2} + 1 = 2^m \cdot k + 2$ divides $2(2^m \pm 1)$, that is,

$$2^{m-1} \cdot k + 1 \mid 2^m \pm 1.$$

This implies $k = 1$, finally leading to a contradiction since

$$2^{m-1} + 1 < 2^m \pm 1 < 2(2^{m-1} + 1)$$

for $m \geq 3$.

02.17. *Show that the sequence*

$$\binom{2002}{2002}, \binom{2003}{2002}, \binom{2004}{2002}, \ldots,$$

considered modulo 2002, is periodic.

Solution: Define

$$x_n^k = \tbinom{n}{k}$$

and note that

$$x_{n+1}^k - x_n^k = \tbinom{n+1}{k} - \tbinom{n}{k} = \tbinom{n}{k-1} = x_n^{k-1}.$$

Let m be any positive integer. We will prove by induction on k that the sequence $\{x_n^k\}_{n=k}^{\infty}$ is periodic modulo m. For $k=1$ it is obvious that $x_n^k = n$ is periodic modulo m with period m. Therefore it will suffice to show that the following is true: *A sequence $\{x_n\}$ is periodic modulo m if its difference sequence, $d_n = x_{n+1} - x_n$, is periodic modulo m.*

Indeed, let t be the period of $\{d_n\}$ and h be the smallest positive integer such that $h(x_t - x_0) \equiv 0 \pmod{m}$. Then

$$\begin{aligned} x_{n+ht} = x_0 + \sum_{j=0}^{n+ht-1} d_j &\equiv x_0 + \sum_{j=0}^{n-1} d_j + h\sum_{j=0}^{t-1} d_j \\ &= x_n + h(x_t - x_0) \equiv x_n \pmod{m} \end{aligned}$$

for all n, so the sequence $\{x_n\}$ is in fact periodic modulo m (with a period dividing ht).

02.18. *Find all integers $n > 1$ such that any prime divisor of $n^6 - 1$ is a divisor of $(n^3-1)(n^2-1)$.*

Answer: Only $n = 2$.

Solution: Consider the equality

$$n^6 - 1 = (n^2 - n + 1)(n+1)(n^3 - 1).$$

The integer $n^2 - n + 1 = n(n-1) + 1$ clearly has an odd divisor p. Then $p \mid n^3 + 1$. Therefore, p does not divide $n^3 - 1$ and consequently $p \mid n^2 - 1$. This implies that p divides $(n^3+1) + (n^2-1) = n^2(n+1)$.

As p does not divide n, we obtain $p \mid n+1$. Also, $p \mid (n^2-1) - (n^2-n+1) = n-2$. From $p \mid n+1$ and $p \mid n-2$ it follows that $p = 3$, so $n^2 - n + 1 = 3^r$ for some positive integer r.

The discriminant of the quadratic $n^2 - n + (1 - 3^r)$ must be a square of an integer, hence

$$1 - 4(1 - 3^r) = 3(4 \cdot 3^{r-1} - 1)$$

must be a square of an integer. Since for $r \geq 2$ the number $4 \cdot 3^{r-1} - 1$ is not divisible by 3, this is possible only if $r = 1$. So $n^2 - n - 2 = 0$ and $n = 2$.

02.19. *Let n be a positive integer. Prove that the equation*

$$x + y + \tfrac{1}{x} + \tfrac{1}{y} = 3n$$

does not have solutions in positive rational numbers.

Solution: Suppose $x = \frac{p}{q}$ and $y = \frac{r}{s}$ satisfy the given equation, where p, q, r, s are positive integers and $\gcd(p,q) = \gcd(r,s) = 1$. We have

$$\frac{p}{q} + \frac{r}{s} + \frac{q}{p} + \frac{s}{r} = 3n,$$

or

$$(p^2+q^2)rs + (r^2+s^2)pq = 3npqrs,$$

so $rs \mid (r^2+s^2)pq$. Since $\gcd(r,s) = 1$, we have $\gcd(r^2+s^2, rs) = 1$ and $rs \mid pq$. Analogously $pq \mid rs$, so $rs = pq$ and hence there are either two or zero integers divisible by 3 among p, q, r, s. Now we have

$$\begin{aligned} (p^2+q^2)rs + (r^2+s^2)rs &= 3n(rs)^2, \\ p^2+q^2+r^2+s^2 &= 3nrs, \end{aligned}$$

but $3nrs \equiv 0 \pmod{3}$ and $p^2+q^2+r^2+s^2$ is congruent to either 1 or 2 modulo 3, a contradiction.

02.20. *Does there exist an infinite non-constant arithmetic progression, each term of which is of the form a^b, where a and b are positive integers with $b \geq 2$?*
Answer: No.
Solution: For an arithmetic progression $a_1, a_2, \ldots$ with difference d the following holds:

$$S_n = \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_{n+1}} = \frac{1}{a_1} + \frac{1}{a_1+d} + \cdots + \frac{1}{a_1+nd}$$
$$\geq \frac{1}{m}\Big(\frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n+1}\Big),$$

where $m = \max(a_1, d)$. Therefore S_n tends to infinity when n increases.

On the other hand, the sum of reciprocals of the powers of a natural number $x \neq 1$ is

$$\frac{1}{x^2} + \frac{1}{x^3} + \cdots = \frac{\frac{1}{x^2}}{1-\frac{1}{x}} = \frac{1}{x(x-1)}\ .$$

Hence, the sum of reciprocals of the terms of the progression required in the problem cannot exceed

$$\frac{1}{1} + \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots = 1 + \Big(\frac{1}{1} - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \cdots\Big) = 2,$$

a contradiction.

Solution 2: Let $a_k = a_0 + dk$, $k = 0, 1, \ldots$. Choose a prime number $p > d$ and set $k' \equiv (p - a_0)d^{-1} \pmod{p^2}$. Then $a_{k'} = a_0 + k'd \equiv p \pmod{p^2}$ and hence, $a_{k'}$ can not be a power of a natural number.

Baltic Way 2003

03.1. *Let $\mathbb{Q}_+$ be the set of positive rational numbers. Find all functions $f\colon \mathbb{Q}_+ \to \mathbb{Q}_+$ which for all $x \in \mathbb{Q}_+$ fulfil*

(1) $f(\frac{1}{x}) = f(x)$
(2) $(1+\frac{1}{x})f(x) = f(x+1)$

Solution: Set $g(x) = \frac{f(x)}{f(1)}$. Function g fulfils (1), (2) and $g(1) = 1$. First we prove that if g exists then it is unique. We prove that g is uniquely defined on $x = \frac{p}{q}$ by induction on $\max(p,q)$. If $\max(p,q) = 1$ then $x = 1$ and $g(1) = 1$. If $p = q$ then $x = 1$ and $g(x)$ is unique. If $p \neq q$ then we can assume (according to (1)) that $p > q$. From (2) we get $g(\frac{p}{q}) = (1+\frac{q}{p-q})g(\frac{p-q}{q})$. The induction assumption and $\max(p,q) > \max(p-q,q) \geq 1$ now give that $g(\frac{p}{q})$ is unique.

Define the function g by $g(\frac{p}{q}) = pq$ where p and q are chosen such that $\gcd(p,q) = 1$. It is easily seen that g fulfils (1), (2) and $g(1) = 1$. All functions fulfilling (1) and (2) are therefore $f(\frac{p}{q}) = apq$, where $\gcd(p,q) = 1$ and $a \in \mathbb{Q}_+$.

03.2. *Prove that any real solution of*

$$x^3 + px + q = 0$$

satisfies the inequality $4qx \leq p^2$.

Solution: Let x_0 be a root of the qubic, then $x^3 + px + q = (x - x_0)(x^2 + ax + b) = x^3 + (a - x_0)x^2 + (b - ax_0)x - bx_0$. So $a = x_0$, $p = b - ax_0 = b - x_0^2$, $-q = bx_0$. Hence $p^2 = b^2 - 2bx_0^2 + x_0^4$. Also $4x_0q = -4x_0^2b$. So $p^2 - 4x_0q = b^2 + 2bx_0^2 + x_0^4 = (b + x_0^2)^2 \geq 0$.

Solution 2: As the equation $x_0x^2 + px + q = 0$ has a root ($x = x_0$), we must have $D \geq 0 \Leftrightarrow p^2 - 4qx_0 \geq 0$. (Also the equation $x^2 + px + qx_0 = 0$ having the root $x = x_0^2$ can be considered.)

03.3. *Let x, y and z be positive real numbers such that $xyz = 1$. Prove that*

$$(1+x)(1+y)(1+z) \geq 2\Big(1 + \sqrt[3]{\frac{y}{x}} + \sqrt[3]{\frac{z}{y}} + \sqrt[3]{\frac{x}{z}}\Big).$$

Solution: Put $a = bx$, $b = cy$ and $c = az$. The given inequality then takes the form

$$\begin{aligned}\Big(1+\frac{a}{b}\Big)\Big(1+\frac{b}{c}\Big)\Big(1+\frac{c}{a}\Big) &\geq 2\Big(1 + \sqrt[3]{\frac{b^2}{ac}} + \sqrt[3]{\frac{c^2}{ab}} + \sqrt[3]{\frac{a^2}{bc}}\Big)\\ &= 2\Big(1 + \frac{a+b+c}{3\sqrt[3]{abc}}\Big).\end{aligned}$$

By the AM-GM inequality we have

$$\begin{aligned}\Big(1+\frac{a}{b}\Big)\Big(1+\frac{b}{c}\Big)\Big(1+\frac{c}{a}\Big) &= \frac{a+b+c}{a} + \frac{a+b+c}{b} + \frac{a+b+c}{c} - 1\\ &\geq 3\Big(\frac{a+b+c}{\sqrt[3]{abc}}\Big) - 1\\ &\geq 2\frac{a+b+c}{\sqrt[3]{abc}} + 3 - 1\\ &= 2\Big(1 + \frac{a+b+c}{\sqrt[3]{abc}}\Big).\end{aligned}$$

Solution 2: Expanding the left side we obtain

$$x + y + z + \tfrac{1}{x} + \tfrac{1}{y} + \tfrac{1}{z} \geq 2\Big(\sqrt[3]{\tfrac{y}{x}} + \sqrt[3]{\tfrac{z}{y}} + \sqrt[3]{\tfrac{x}{z}}\Big).$$

As $\sqrt[3]{\frac{y}{x}} \leq \frac{1}{3}\big(y + \frac{1}{x} + 1\big)$ etc., it suffices to prove that

$$x + y + z + \tfrac{1}{x} + \tfrac{1}{y} + \tfrac{1}{z} \geq \tfrac{2}{3}\big(x + y + z + \tfrac{1}{x} + \tfrac{1}{y} + \tfrac{1}{z}\big) + 2,$$

which follows from $a + \frac{1}{a} \geq 2$.

03.4. *Let a, b, c be positive real numbers. Prove that*

$$\frac{2a}{a^2 + bc} + \frac{2b}{b^2 + ca} + \frac{2c}{c^2 + ab} \leq \frac{a}{bc} + \frac{b}{ca} + \frac{c}{ab}.$$

Solution: First we prove that

$$\frac{2a}{a^2 + bc} \leq \frac{1}{2}\Big(\frac{1}{b} + \frac{1}{c}\Big),$$

which is equivalent to $0 \leq b(a - c)^2 + c(a - b)^2$, and therefore holds true. Now we turn to the inequality

$$\frac{1}{b} + \frac{1}{c} \leq \frac{1}{2}\Big(\frac{2a}{bc} + \frac{b}{ca} + \frac{c}{ab}\Big),$$

which by multiplying by $2abc$ is seen to be equivalent to $0 \leq (a - b)^2 + (a - c)^2$. Hence we have proved that

$$\frac{2a}{a^2 + bc} \leq \frac{1}{4}\Big(\frac{2a}{bc} + \frac{b}{ca} + \frac{c}{ab}\Big).$$

Analogously we have

$$\frac{2b}{b^2 + ca} \leq \frac{1}{4}\Big(\frac{2b}{ca} + \frac{c}{ab} + \frac{a}{bc}\Big),$$
$$\frac{2c}{c^2 + ab} \leq \frac{1}{4}\Big(\frac{2c}{ab} + \frac{a}{bc} + \frac{b}{ca}\Big)$$

and it suffices to sum the above three inequalities.

Solution 2: As $a^2 + bc \geq 2a\sqrt{bc}$ etc., it is sufficient to prove that

$$\frac{1}{\sqrt{bc}} + \frac{1}{\sqrt{ac}} + \frac{1}{\sqrt{ab}} \leq \frac{a}{bc} + \frac{b}{ca} + \frac{c}{ab},$$

which can be obtained by "inserting" $\frac{1}{a} + \frac{1}{b} + \frac{1}{c}$ between the left side and the right side.

03.5. *A sequence (a_n) is defined as follows: $a_1 = \sqrt{2}$, $a_2 = 2$, and $a_{n+1} = a_n a_{n-1}^2$ for $n \geq 2$. Prove that for every $n \geq 1$ we have*

$$(1 + a_1)(1 + a_2)\cdots(1 + a_n) < (2 + \sqrt{2})a_1 a_2 \cdots a_n.$$

Solution: First we prove inductively that for $n \geq 1$, $a_n = 2^{2^{n-2}}$. We have $a_1 = 2^{2^{-1}}$, $a_2 = 2^{2^0}$ and

$$a_{n+1} = 2^{2^{n-2}} \cdot (2^{2^{n-3}})^2 = 2^{2^{n-2}} \cdot 2^{2^{n-2}} = 2^{2^{n-1}}.$$

Since $1 + a_1 = 1 + \sqrt{2}$, we must prove, that

$$(1 + a_2)(1 + a_3) \cdots (1 + a_n) < 2a_2a_3 \cdots a_n.$$

The right-hand side is equal to

$$2^{1+2^0+2^1+\cdots+2^{n-2}} = 2^{2^{n-1}}$$

and the left-hand side

$$\begin{aligned}(1 + 2^{2^0})&(1 + 2^{2^1}) \cdots (1 + 2^{2^{n-2}})\\ &= 1 + 2^{2^0} + 2^{2^1} + 2^{2^0+2^1} + 2^{2^2} + \cdots + 2^{2^0+2^1+\cdots+2^{n-2}}\\ &= 1 + 2 + 2^2 + 2^3 + \cdots + 2^{2^{n-1}-1}\\ &= 2^{2^{n-1}} - 1.\end{aligned}$$

The proof is complete.

03.6. *Let $n \geq 2$ and $d \geq 1$ be integers with $d \mid n$, and let $x_1, x_2, \ldots, x_n$ be real numbers such that $x_1 + x_2 + \cdots + x_n = 0$. Prove that there are at least $\binom{n-1}{d-1}$ choices of d indices $1 \leq i_1 < i_2 < \cdots < i_d \leq n$ such that $x_{i_1} + x_{i_2} + \cdots + x_{i_d} \geq 0$.*

Solution: Put $m = n/d$ and $[n] = \{1, 2, \ldots, n\}$, and consider all partitions $[n] = A_1 \cup A_2 \cup \cdots \cup A_m$ of $[n]$ into d-element subsets A_i, $i = 1, 2, \ldots, m$. The number of such partitions is denoted by t. Clearly, there are exactly $\binom{n}{d}$ d-element subsets of $[n]$ each of which occurs in the same number of partitions. Hence, every $A \subseteq [n]$ with $|A| = d$ occurs in exactly $s := tm/\binom{n}{d}$ partitions. On the other hand, every partition contains at least one d-element set A such that $\sum_{i \in A} x_i \geq 0$. Consequently, the total number of sets with this property is at least $t/s = \binom{n}{d}/m = \frac{d}{n}\binom{n}{d} = \binom{n-1}{d-1}$.

03.7. *Let X be a subset of $\{1, 2, 3, \ldots, 10000\}$ with the following property: If $a, b \in X$, $a \neq b$, then $a \cdot b \notin X$. What is the maximal number of elements in X?*

Answer: 9901.

Solution: If $X = \{100, 101, 102, \ldots, 9999, 10000\}$, then for any two selected a and b, $a \neq b$, $a \cdot b \geq 100 \cdot 101 > 10000$, so $a \cdot b \notin X$. So X may have 9901 elements.

Suppose that $x_1 < x_2 < \cdots < x_k$ are all elements of X that are less than 100. If there are none of them, no more than 9901 numbers can be in the set X. Otherwise, if $x_1 = 1$ no other number can be in the set X, so suppose $x_1 > 1$ and consider the pairs

$$\begin{gathered}200 - x_1,\ (200 - x_1) \cdot x_1\\ 200 - x_2,\ (200 - x_2) \cdot x_2\\ \vdots\\ 200 - x_k,\ (200 - x_k) \cdot x_k\end{gathered}$$

Clearly $x_1 < x_2 < \cdots < x_k < 100 < 200 - x_k < 200 - x_{k-1} < \cdots < 200 - x_2 < 200 - x_1 < 200 < (200 - x_1) \cdot x_1 < (200 - x_2) \cdot x_2 < \cdots < (200 - x_k) \cdot x_k$. So all numbers in these pairs are different and greater than 100. So at most one from each pair is in the set X. Therefore, there are at least k numbers greater than 100 and $99 - k$ numbers less than 100 that are not in the set X, together at least 99 numbers out of 10000 not being in the set X.

03.8. *There are* 2003 *pieces of candy on a table. Two players alternately make moves. A move consists of eating one candy or half of the candies on the table (the "lesser half" if there is an odd number of candies); at least one candy must be eaten at each move. The loser is the one who eats the last candy. Which player – the first or the second – has a winning strategy?*

Answer: The second.

Solution: Let us prove inductively that for $2n$ pieces of candy the first player has a winning strategy. For $n = 1$ it is obvious. Suppose it is true for $2n$ pieces, and let's consider $2n + 2$ pieces. If for $2n + 1$ pieces the second is the winner, then the first eats 1 piece and becomes the second in the game starting with $2n + 1$ pieces. So suppose that for $2n + 1$ pieces the first is the winner. His winning move for $2n + 1$ is not eating 1 piece (according to the inductive assumption). So his winning move is to eat n pieces, leaving the second with $n + 1$ pieces, when the second must lose. But the first can leave the second with $n + 1$ pieces from the starting position with $2n + 2$ pieces, eating $n + 1$ pieces; so $2n + 2$ is a winning position for the first.

Now if there are 2003 pieces of candy on the table, the first must eat either 1 or 1001 candies, leaving an even number of candies on the table. So the second player will be the first player in a game with even number of candies and therefore has a winning strategy.

In general, if there is an odd number N of candies, write $N = 2^m r + 1$, where r is odd. Then the first player wins if m is even, and the second player wins if m is odd: At each move, the player must avoid leaving the other with an even number of candies, so he must eat half of the candies. But this means that the number of candies descend as $2^m r + 1, 2^{m-1} r + 1, \dots, 2r + 1, r + 1$, and eventually there is an even number of candies.

03.9. *It is known that n is a positive integer, $n \leq 144$. Ten questions of type "Is n smaller than a?" are allowed. Answers are given with a delay: The answer to the i'th question is given only after the $(i + 1)$'st question is asked, $i = 1, 2, \dots, 9$. The answer to the tenth question is given immediately after it is asked. Find a strategy for identifying n.*

Solution: Let the Fibonacci numbers be denoted $F_0 = 1$, $F_1 = 2$, $F_2 = 3$ etc. Then $F_{10} = 144$. We will prove by induction on k that using k questions subject to the conditions of the problem, it is possible to determine any positive integer $n \leq F_k$. First, for $k = 0$ it is trivial, since without asking we know that $n = 1$. For $k = 1$, we simply ask if n is smaller than 2. For $k = 2$, we ask if n is smaller than 3 and if n is smaller than 2; from the two answers we can determine n.

Now, in general, our first two questions will always be "Is n smaller than $F_{k-1} + 1$?" and "Is n smaller than $F_{k-2} + 1$". We then receive the answer to the first question. As long as we receive affirmative answers to the $i - 1$'st question, the $i + 1$'st question will be "Is n smaller than $F_{k-(i+1)} + 1$?". If at any point, say after asking the j'th question, we receive a negative answer to the $j - 1$'st question, we then know that $F_{k-(j-1)} + 1 \leq n \leq F_{k-(j-2)}$, so n is one of $F_{k-(j-2)} - F_{k-(j-1)} = F_{k-j}$ consecutive integers, and by induction we may determine n using the remaining $k - j$ questions. Otherwise, we receive affirmative answers to all the questions, the last being "Is n smaller than $F_{k-k} + 1 = 2$?"; so $n = 1$ in that case.

03.10. *A* lattice point *in the plane is a point whose coordinates are both integral. The* centroid *of four points (x_i, y_i), $i = 1, 2, 3, 4$, is the point $(\frac{x_1+x_2+x_3+x_4}{4}, \frac{y_1+y_2+y_3+y_4}{4})$. Let n be the largest natural number with the following property: There are n distinct lattice points in the plane such that the centroid of any four of them is not a lattice point. Prove that $n = 12$.*

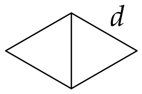
Solution: To prove $n \geq 12$, we have to show that there are 12 lattice points (x_i, y_i), $i = 1, 2, \dots, 12$, such that no four determine a lattice point centroid. This is guaranteed if we just choose the points such that $x_i \equiv 0 \pmod 4$ for $i = 1, \dots, 6$, $x_i \equiv 1 \pmod 4$ for $i = 7, \dots, 12$, $y_i \equiv 0 \pmod 4$ for $i = 1, 2, 3, 10, 11, 12$, $y_i \equiv 1 \pmod 4$ for $i = 4, \dots, 9$.

Now let P_i, $i = 1, 2, \dots, 13$, be lattice points. We have to show that some four of them determine a lattice point centroid. First observe that, by the Pigeonhole Principle, among any five of the points we find two such that their x-coordinates as well as their y-coordinates have the same parity. Consequently, among any five of the points there are two whose midpoint is a lattice point. Iterated application of this observation implies that among the 13 points in question we find five disjoint pairs of points whose midpoint is a lattice point. Among these five midpoints we again find two, say M and M', such that their midpoint C is a lattice point. Finally, if M and M' are the midpoints of P_iP_j and P_kP_ℓ, respectively, $\{i, j, k, \ell\} \subseteq \{1, 2, \dots, 13\}$, then C is the centroid of P_i, P_j, P_k, P_ℓ.

03.11. *Is it possible to select* 1000 *points in a plane so that at least* 6000 *distances between two of them are equal?*

Answer: Yes.

Solution: Let's start with configuration of 4 points and 5 distances equal to d, like in this figure:

(α) 

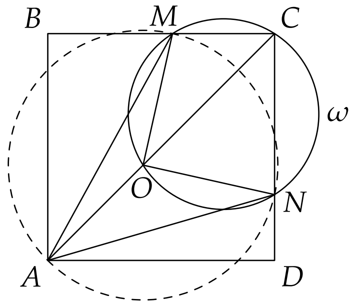
Now take (α) and two copies of it obtainable by parallel shifts along vectors $\vec{a}$ and $\vec{b}$, $|\vec{a}| = |\vec{b}| = d$ and $\angle(\vec{a}, \vec{b}) = 60^\circ$. Vectors $\vec{a}$ and $\vec{b}$ should be chosen so that no two vertices of (α) and of the two copies coincide. We get $3 \cdot 4 = 12$ points and $3 \cdot 5 + 12 = 27$ distances. Proceeding in the same way, we get gradually

- $3 \cdot 12 = 36$ points and $3 \cdot 27 + 36 = 117$ distances;
- $3 \cdot 36 = 108$ points and $3 \cdot 117 + 108 = 459$ distances;
- $3 \cdot 108 = 324$ points and $3 \cdot 459 + 324 = 1701$ distances;
- $3 \cdot 324 = 972$ points and $3 \cdot 1701 + 972 = 6075$ distances.

03.12. *Let $ABCD$ be a square. Let M be an inner point on side BC and N be an inner point on side CD with $\angle MAN = 45^\circ$. Prove that the circumcentre of AMN lies on AC.*

Solution: Draw a circle ω through M, C, N; let it intersect AC at O. We claim that O is the circumcentre of AMN.

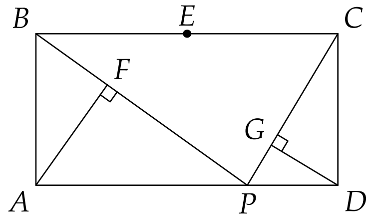
Clearly $\angle MON = 180^\circ - \angle MCN = 90^\circ$. If the radius of ω is R, then $OM = 2R \sin 45^\circ = R\sqrt{2}$; similarly $ON = R\sqrt{2}$. Hence we get that $OM = ON$. Then the circle with centre O and radius $R\sqrt{2}$ will pass through A, since $\angle MAN = \frac{1}{2}\angle MON$.



03.13. *Let $ABCD$ be a rectangle and $BC = 2 \cdot AB$. Let E be the midpoint of BC and P an arbitrary inner point of AD. Let F and G be the feet of perpendiculars drawn correspondingly from A to BP and from D to CP. Prove that the points E, F, P, G are concyclic.*

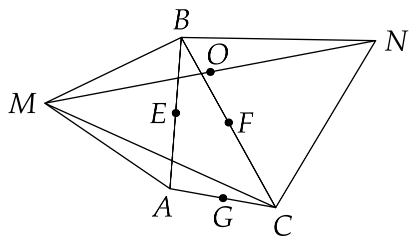
Solution: From rectangular triangle BAP we have $BP \cdot BF = AB^2 = BE^2$. Therefore the circumference through F and P touching the line BC between B and C touches it at E.

Analogously, the circumference through P and G touching the line BC between B and C touches it at E. But there is only one circumference touching BC at E and passing through P.

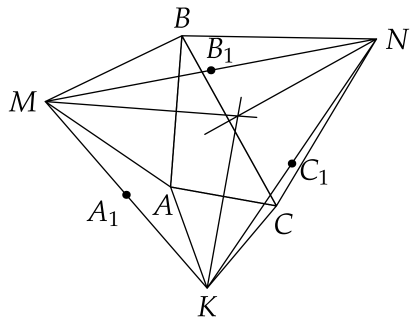


03.14. *Let ABC be an arbitrary triangle and AMB, BNC, CKA regular triangles outward of ABC. Through the midpoint of MN a perpendicular to AC is constructed; similarly through the midpoints of NK resp. KM perpendiculars to AB resp. BC are constructed. Prove that these three perpendiculars intersect at the same point.*

Solution: Let O be the midpoint of MN, and let E and F be the midpoints of AB and BC, respectively. As triangle MBC transforms into triangle ABN when rotated 60° around B we get $MC = AN$ (it is also a well-known fact). Considering now the quadrangles $AMBN$ and $CMBN$ we get $OE = OF$ (from Eiler's formula $a^2 + b^2 + c^2 + d^2 = e^2 + f^2 + 4 \cdot PQ^2$ or otherwise). As $EF \parallel AC$ we get from this that the perpendicular to AC through O passes through the circumcentre of EFG, as it is the perpendicular bisector of EF. The same holds for the other two perpendiculars.



First solution

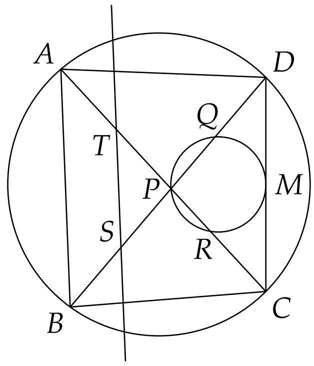


Second solution

Solution 2: Let us denote the midpoints of the segments MN, NK, KM by B_1, C_1, A_1, respectively. It is easy to see that triangle $A_1B_1C_1$ is homothetic to triangle NKM via the homothety centered at the intersection of the medians of triangle NMK and dilation $-\frac{1}{2}$. The perpendiculars through M, N, K to AB, BC, CA, respectively, are also the perpendicular bisectors of these sides, so they intersect in the circumcentre of triangle ABC. The desired result follows now from the homothety, and we find that that the common point of intersection is the circumcentre of the image of triangle ABC under the homothety; that is, the circumcentre of the triangle with vertices the midpoints of the sides AB, BC, CA.

03.15. *Let P be the intersection point of the diagonals AC and BD in a cyclic quadrilateral. A circle through P touches the side CD in the midpoint M of this side and intersects the segments BD and AC in the points Q and R, respectively. Let S be a point on the segment BD such that $BS = DQ$. The parallel to AB through S intersects AC at T. Prove that $AT = RC$.*

Solution: With reference to the figure below we have $CR \cdot CP = DQ \cdot DP = CM^2 = DM^2$, which is equivalent to $RC = \frac{DQ \cdot DP}{CP}$. We also have $\frac{AT}{BS} = \frac{AP}{BP} = \frac{AT}{DQ}$, so $AT = \frac{AP \cdot DQ}{BP}$. Since $ABCD$ is cyclic the result now comes from the fact that $DP \cdot BP = AP \cdot CP$ (due to a well-known theorem).



03.16. *Find all pairs of positive integers (a, b) such that $a - b$ is a prime and ab is a perfect square.*

Answer: Pairs $(a, b) = ((\frac{p+1}{2})^2, (\frac{p-1}{2})^2)$, where p is a prime greater than 2.

Solution: Let p be a prime such that $a - b = p$ and let $ab = k^2$. Insert $a = b + p$ in the equation $ab = k^2$. Then

$$k^2 = (b + p)b = (b + \frac{p}{2})^2 - \frac{p^2}{4}$$

which is equivalent to

$$p^2 = (2b + p)^2 - 4k^2 = (2b + p + 2k)(2b + p - 2k).$$

Since $2b + p + 2k > 2b + p - 2k$ and p is a prime, we conclude $2b + p + 2k = p^2$ and $2b + p - 2k = 1$. By adding these equations we get $2b + p = \frac{p^2+1}{2}$ and then $b = (\frac{p-1}{2})^2$, so $a = b + p = (\frac{p+1}{2})^2$. By checking we conclude that all the solutions are $(a, b) = ((\frac{p+1}{2})^2, (\frac{p-1}{2})^2)$ with p a prime greater than 2.

Solution 2: Let p be a prime such that $a - b = p$ and let $ab = k^2$. We have $(b + p)b = k^2$, so $\gcd(b, b + p) = \gcd(b, p)$ is equal either to 1 or p. If $\gcd(b, b + p) = p$, let $b = b_1 p$. Then $p^2 b_1(b_1 + 1) = k^2$, $b_1(b_1 + 1) = m^2$, but this equation has no solutions.

Hence $\gcd(b, b + p) = 1$, and

$$b = u^2 \qquad\qquad b + p = v^2$$

so that $p = v^2 - u^2 = (v + u)(v - u)$. This in turn implies that $v - u = 1$ and $v + u = p$, from which we finally obtain $a = \left(\frac{p+1}{2}\right)^2$, $b = \left(\frac{p-1}{2}\right)^2$, where p must be an odd prime.

03.17. *All the positive divisors of a positive integer n are stored into an array in increasing order. Mary has to write a program which decides for an arbitrarily chosen divisor $d > 1$ whether it is a prime. Let n have k divisors not greater than d. Mary claims that it suffices to check divisibility of d by the first $\lceil k/2 \rceil$ divisors of n: If a divisor of d greater than 1 is found among them, then d is composite, otherwise d is prime. Is Mary right?*

Answer: Yes, Mary is right.

Solution: Let $d > 1$ be a divisor of n. Suppose Mary's program outputs "composite" for d. That means it has found a divisor of d greater than 1. Since $d > 1$, the array contains at least 2 divisors of d, namely 1 and d. Thus Mary's program does not check divisibility of d by d (the first half gets complete before reaching d) which means that the divisor found lays strictly between 1 and d. Hence d is composite indeed.

Suppose now d being composite. Let p be its smallest prime divisor; then $\frac{d}{p} \geq p$ or, equivalently, $d \geq p^2$. As p is a divisor of n, it occurs in the array. Let $a_1, \ldots, a_k$ all divisors of n smaller than p. Then $pa_1, \ldots, pa_k$ are less than p^2 and hence less than d. As $a_1, \ldots, a_k$ are all relatively prime with p, all the numbers $pa_1, \ldots, pa_k$ divide n. The numbers $a_1, \ldots, a_k, pa_1, \ldots, pa_k$ are pairwise different by construction. Thus there are at least $2k+1$ divisors of n not greater than d. So Mary's program checks divisibility of d by at least $k+1$ smallest divisors of n, among which it finds p, and outputs "composite".

03.18. *Every integer is coloured with exactly one of the colours* BLUE, GREEN, RED, YELLOW. *Can this be done in such a way that if a, b, c, d are not all* 0 *and have the same colour, then $3a - 2b \neq 2c - 3d$?*

Answer: Yes.

Solution: A colouring with the required property can be defined as follows. For a non-zero integer k let k^* be the integer uniquely defined by $k = 5^m \cdot k^*$, where m is a nonnegative integer and $5 \nmid k^*$. We also define $0^* = 0$. Two non-zero integers k_1, k_2 receive the same colour if and only if $k_1^* \equiv k_2^* \pmod 5$; we assign 0 any colour.

Assume a, b, c, d has the same colour and that $3a - 2b = 2c - 3d$, which we rewrite as $3a - 2b - 2c + 3d = 0$. Dividing both sides by the largest power of 5 which simultaneously divides a, b, c, d (this makes sense since not all of a, b, c, d are 0), we obtain

$$3 \cdot 5^A \cdot a^* - 2 \cdot 5^B \cdot b^* - 2 \cdot 5^C \cdot c^* + 3 \cdot 5^D \cdot d^* = 0,$$

where A, B, C, D are nonnegative integers at least one of which is equal to 0. The above equality implies

$$3(5^A \cdot a^* + 5^B \cdot b^* + 5^C \cdot c^* + 5^D \cdot d^*) \equiv 0 \pmod 5.$$

Assume a, b, c, d are all non-zero. Then $a^* \equiv b^* \equiv c^* \equiv d^* \not\equiv 0 \pmod 5$. This implies

$$5^A + 5^B + 5^C + 5^D \equiv 0 \pmod 5 \tag{03.15}$$

which is impossible since at least one of the numbers A, B, C, D is equal to 0. If one or more of a, b, c, d are 0, we simply omit the corresponding terms from (03.15), and the same conclusion holds.

03.19. *Let a and b be positive integers. Prove that if $a^3 + b^3$ is the square of an integer, then $a + b$ is not a product of two different prime numbers.*

Solution: Suppose $a + b = pq$, where $p \neq q$ are two prime numbers. We may assume that $p \neq 3$. Since

$$a^3 + b^3 = (a+b)(a^2 - ab + b^2)$$

is a square, the number $a^2 - ab + b^2 = (a+b)^2 - 3ab$ must be divisible by p and q, whence $3ab$ must be divisible by p and q. But $p \neq 3$, so $p \mid a$ or $p \mid b$; but $p \mid a + b$, so $p \mid a$ and $p \mid b$. Write $a = pk$, $b = p\ell$ for some integers k, ℓ. Notice that $q = 3$, since otherwise, repeating the above argument, we would have $q \mid a$, $q \mid b$ and $a + b > pq$). So we have

$$3p = a + b = p(k + \ell)$$

and we conclude that $a = p$, $b = 2p$ or $a = 2p$, $b = p$. Then $a^3 + b^3 = 9p^3$ is obviously not a square, a contradiction.

03.20. *Let n be a positive integer such that the sum of all the positive divisors of n (except n) plus the number of these divisors is equal to n. Prove that $n = 2m^2$ for some integer m.*
Solution: Let $t_1 < t_2 < \cdots < t_s$ be all positive odd divisors of n, and let 2^k be the maximal power of 2 that divides n. Then the full list of divisors of n is the following:

$$t_1, \ldots, t_s, 2t_1, \ldots, 2t_s, \ldots, 2^k t_1, \ldots, 2^k t_s.$$

Hence,

$$2n = (2^{k+1} - 1)(t_1 + t_2 + \cdots + t_s) + (k+1)s - 1.$$

The right-hand side can be even only if both k and s are odd. In this case the number $n/2^k$ has an odd number of divisors and therefore it is equal to a perfect square r^2. Writing $k = 2a + 1$, we have $n = 2^k r^2 = 2(2^a r)^2$.

Baltic Way 2004

04.1. *Given a sequence $a_1, a_2, a_3, \ldots$ of non-negative real numbers satisfying the conditions*

(1) $a_n + a_{2n} \geq 3n$

(2) $a_{n+1} + n \leq 2\sqrt{a_n \cdot (n+1)}$

for all indices $n = 1, 2 \ldots$.

(a) *Prove that the inequality $a_n \geq n$ holds for every $n \in \mathbb{N}$.*

(b) *Give an example of such a sequence.*

Solution: (a) Note that the inequality

$$\frac{a_{n+1} + n}{2} \geq \sqrt{a_{n+1} \cdot n}$$

holds, which together with the second condition of the problem gives

$$\sqrt{a_{n+1} \cdot n} \leq \sqrt{a_n \cdot (n+1)}.$$

This inequality simplifies to

$$\frac{a_{n+1}}{a_n} \leq \frac{n+1}{n}.$$

Now, using the last inequality for the index n replaced by $n, n+1, \ldots, 2n-1$ and multiplying the results, we obtain

$$\frac{a_{2n}}{a_n} \leq \frac{2n}{n} = 2$$

or $2a_n \geq a_{2n}$. Taking into account the first condition of the problem, we have

$$3a_n = a_n + 2a_n \geq a_n + a_{2n} \geq 3n$$

which implies $a_n \geq n$. (b) The sequence defined by $a_n = n + 1$ satisfies all the conditions of the problem.

04.2. *Let $P(x)$ be a polynomial with non-negative coefficients. Prove that if $P(\frac{1}{x})P(x) \geq 1$ for $x = 1$, then the same inequality holds for each positive x.*

Solution: For $x > 0$ we have $P(x) > 0$ (because at least one coefficient is non-zero). From the given condition we have $(P(1))^2 \geq 1$. Further, let's denote $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$. Then

$$\begin{aligned} P(x)P(\tfrac{1}{x}) &= (a_n x^n + \cdots + a_0)(a_n x^{-n} + \cdots + a_0) \\ &= \sum_{i=0}^{n} a_i^2 + \sum_{i=1}^{n}\sum_{j=0}^{i-1}(a_{i-j}a_j)(x^i + x^{-i}) \\ &\geq \sum_{i=0}^{n} a_i^2 + 2\sum_{i>j} a_i a_j \\ &= (P(1))^2 \geq 1. \end{aligned}$$

04.3. *Let p, q, r be positive real numbers and $n \in \mathbb{N}$. Show that if $pqr = 1$, then*

$$\frac{1}{p^n + q^n + 1} + \frac{1}{q^n + r^n + 1} + \frac{1}{r^n + p^n + 1} \le 1.$$

Solution: The key idea is to deal with the case $n = 3$. Put $a = p^{n/3}$, $b = q^{n/3}$, and $c = r^{n/3}$, so $abc = (pqr)^{n/3} = 1$ and

$$\tfrac{1}{p^n+q^n+1} + \tfrac{1}{q^n+r^n+1} + \tfrac{1}{r^n+p^n+1} = \tfrac{1}{a^3+b^3+1} + \tfrac{1}{b^3+c^3+1} + \tfrac{1}{c^3+a^3+1}.$$

Now

$$\tfrac{1}{a^3+b^3+1} = \tfrac{1}{(a+b)(a^2-ab+b^2)+1} = \tfrac{1}{(a+b)((a-b)^2+ab)+1} \le \tfrac{1}{(a+b)ab+1}.$$

Since $ab = c^{-1}$,

$$\tfrac{1}{a^3+b^3+1} \le \tfrac{1}{(a+b)ab+1} = \tfrac{c}{a+b+c}.$$

Similarly we obtain

$$\tfrac{1}{b^3+c^3+1} \le \tfrac{a}{a+b+c} \qquad \text{and} \qquad \tfrac{1}{c^3+a^3+1} \le \tfrac{b}{a+b+c}.$$

Hence

$$\tfrac{1}{a^3+b^3+1} + \tfrac{1}{b^3+c^3+1} + \tfrac{1}{c^3+a^3+1} \le \tfrac{c}{a+b+c} + \tfrac{a}{a+b+c} + \tfrac{b}{a+b+c} = 1,$$

which was to be shown.

04.4. *Let $x_1, x_2, \dots, x_n$ be real numbers with arithmetic mean X. Prove that there is a positive integer K such that the arithmetic mean of each of the lists $\{x_1, x_2, \dots, x_K\}$, $\{x_2, x_3, \dots, x_K\}$, $\dots$, $\{x_{K-1}, x_K\}$, $\{x_K\}$ is not greater than X.*

Solution: Suppose the conclusion is false. This means that for every $K \in \{1, 2, \dots, n\}$, there exists a $k \le K$ such that the arithmetic mean of $x_k, x_{k+1}, \dots, x_K$ exceeds X. We now define a decreasing sequence $b_1 \ge a_1 > a_1 - 1 = b_2 \ge a_2 > \cdots$ as follows: Put $b_1 = n$, and for each i, let a_i be the largest largest $k \le b_i$ such that the arithmetic mean of $x_{a_i}, \dots, x_{b_i}$ exceeds X; then put $b_{i+1} = a_i - 1$ and repeat. Clearly for some m, $a_m = 1$. Now, by construction, each of the sets $\{x_{a_m}, \dots, x_{b_m}\}$, $\{x_{a_{m-1}}, \dots, x_{b_{m-1}}\}$, $\dots$, $\{x_{a_1}, \dots, x_{b_1}\}$ has arithmetic mean strictly greater than X, but then the union $\{x_1, x_2, \dots, x_n\}$ of these sets has arithmetic mean strictly greater than X; a contradiction.

04.5. *Determine the range of the function f defined for integers k by*

$$f(k) = (k)_3 + (2k)_5 + (3k)_7 - 6k,$$

where $(k)_{2n+1}$ denotes the multiple of $2n + 1$ closest to k.

Solution: For odd n we have

$$(k)_n = k + \frac{n-1}{2} - \big[k + \frac{n-1}{2}\big]_n,$$

where $[m]_n$ denotes the principal remainder of m modulo n. Hence we get

$$f(k) = 6 - [k+1]_3 - [2k+2]_5 - [3k+3]_7.$$

The condition that the principal remainders take the values a, b and c, respectively, may be written

$$\begin{aligned} k+1 &\equiv a \pmod 3, \\ 2k+2 &\equiv b \pmod 5, \\ 3k+3 &\equiv c \pmod 7 \end{aligned}$$

or

$$\begin{aligned} k &\equiv a-1 \pmod 3, \\ k &\equiv -2b-1 \pmod 5, \\ k &\equiv -2c-1 \pmod 7. \end{aligned}$$

By the Chinese Remainder Theorem, these congruences have a solution for any set of a, b, c. Hence f takes all the integer values between $6-2-4-6=-6$ and $6-0-0-0=6$. (In fact, this proof also shows that f is periodic with period $3 \cdot 5 \cdot 7 = 105$.)

04.6. *A positive integer is written on each of the six faces of a cube. For each vertex of the cube we compute the product of the numbers on the three adjacent faces. The sum of these products is* 1001*. What is the sum of the six numbers on the faces?*

Solution: Let the numbers on the faces be a_1, a_2, b_1, b_2, c_1, c_2, placed so that a_1 and a_2 are on opposite faces etc. Then the sum of the eight products is equal to

$$(a_1+a_2)(b_1+b_2)(c_1+c_2) = 1001 = 7 \cdot 11 \cdot 13.$$

Hence the sum of the numbers on the faces is $a_1+a_2+b_1+b_2+c_1+c_2 = 7+11+13 = 31$.

04.7. *Find all sets X consisting of at least two positive integers such that for every pair $m, n \in X$, where $n > m$, there exists $k \in X$ such that $n = mk^2$.*

Answer: The sets $\{m, m^3\}$, where $m > 1$.

Solution: Let X be a set satisfying the condition of the problem and let $n > m$ be the two smallest elements in the set X. There has to exist a $k \in X$ so that $n = mk^2$, but as $m \le k \le n$, either $k = n$ or $k = m$. The first case gives $m = n = 1$, a contradiction; the second case implies $n = m^3$ with $m > 1$.

Suppose there exists a third smallest element $q \in X$. Then there also exists $k_0 \in X$, such that $q = mk_0^2$. We have $q > k_0 \ge m$, but $k_0 = m$ would imply $q = n$, thus $k_0 = n = m^3$ and $q = m^7$. Now for q and n there has to exist $k_1 \in X$ such that $q = nk_1^2$, which gives $k_1 = m^2$. Since $m^2 \notin X$, we have a contradiction.

Thus we see that the only possible sets are those of the form $\{m, m^3\}$ with $m > 1$, and these are easily seen to satisfy the conditions of the problem.

04.8. *Let f be a non-constant polynomial with integer coefficients. Prove that there is an integer n such that $f(n)$ has at least* 2004 *distinct prime factors.*

Solution: Suppose the contrary. Choose an integer n_0 so that $f(n_0)$ has the highest number of prime factors. By translating the polynomial we may assume $n_0 = 0$. Setting $k = f(0)$, we have $f(wk^2) \equiv k \pmod{k^2}$, or $f(wk^2) = ak^2 + k = (ak+1)k$. Since $\gcd(ak+1, k) = 1$ and k alone achieves the highest number of prime factors of f, we must have $ak + 1 = \pm 1$. This cannot happen for every w since f is non-constant, so we have a contradiction.

04.9. *A set S of $n-1$ natural numbers is given ($n \geq 3$). There exists at least two elements in this set whose difference is not divisible by n. Prove that it is possible to choose a non-empty subset of S so that the sum of its elements is divisible by n.*
Solution: Suppose to the contrary that there exists a set $X = \{a_1, a_2, \ldots, a_{n-1}\}$ violating the statement of the problem, and let $a_{n-2} \not\equiv a_{n-1} \pmod{n}$. Denote $S_i = a_1 + a_2 + \cdots + a_i$, $i = 1, \ldots, n-1$. The conditions of the problem imply that all the numbers S_i must give different remainders when divided by n. Indeed, if for some $j < k$ we had $S_j \equiv S_k \pmod{n}$, then $a_{j+1} + a_{j+2} + \cdots + a_k = S_k - S_j \equiv 0 \pmod{n}$. Consider now the sum $S' = S_{n-3} + a_{n-1}$. We see that S' can not be congruent to any of the sums S_i (for $i \neq n-2$ the above argument works and for $i = n-2$ we use the assumption $a_{n-2} \not\equiv a_{n-1} \pmod{n}$). Thus we have n sums that give pairwise different remainders when divided by n, consequently one of them has to give the remainder 0, a contradiction.

04.10. *Is there an infinite sequence of prime numbers $p_1, p_2, \ldots$ such that $|p_{n+1} - 2p_n| = 1$ for each $n \in \mathbb{N}$?*
Answer: No, there is no such sequence.
Solution: Suppose the contrary. Clearly $p_3 > 3$. There are two possibilities: If $p_3 \equiv 1 \pmod{3}$ then necessarily $p_4 = 2p_3 - 1$ (otherwise $p_4 \equiv 0 \pmod{3}$), so $p_4 \equiv 1 \pmod{3}$. Analogously $p_5 = 2p_4 - 1$, $p_6 = 2p_5 - 1$ etc. By an easy induction we have

$$p_{n+1} - 1 = 2^{n-2}(p_3 - 1), \quad n = 3, 4, 5, \ldots.$$

If we set $n = p_3 + 1$ we have $p_{p_3+2} - 1 = 2^{p_3-1}(p_3 - 1)$, from which

$$p_{p_3+2} \equiv 1 + 1 \cdot (p_3 - 1) = p_3 \equiv 0 \pmod{p_3},$$

a contradiction. The case $p_3 \equiv 2 \pmod{3}$ is treated analogously.

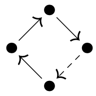
04.11. *An $m \times n$ table is given, in each cell of which a number $+1$ or -1 is written. It is known that initially exactly one -1 is in the table, all the other numbers being $+1$. During a move, it is allowed to choose any cell containing -1, replace this -1 by 0, and simultaneously multiply all the numbers in the neighboring cells by -1 (we say that two cells are neighboring if they have a common side). Find all (m, n) for which using such moves one can obtain the table containing zeroes only, regardless of the cell in which the initial -1 stands.*
Answer: Those (m, n) for which at least one of m, n is odd.
Solution: Let us erase a unit segment which is the common side of any two cells in which two zeroes appear. If the final table consists of zeroes only, all the unit segments (except those which belong to the boundary of the table) are erased. We must erase a total of

$$m(n-1) + n(m-1) = 2mn - m - n$$

such unit segments.

On the other hand, in order to obtain 0 in a cell with initial $+1$ one must first obtain -1 in this cell, that is, the sign of the number in this cell must change an odd number of times (namely, 1 or 3). Hence, any cell with -1 (except the initial one) has an odd number of neighboring zeroes. So, any time we replace -1 by 0 we erase an odd number of unit segments. That is, the total number of unit segments is congruent modulo 2 to the initial number of $+1$'s in the table. Therefore $2mn - m - n \equiv mn - 1 \pmod{2}$, implying that $(m-1)(n-1) \equiv 0 \pmod{2}$, so at least one of m, n is odd.

It remains to show that if, for example, n is odd, we can obtain a zero table. First, if -1 is in the i'th row, we may easily make the i'th row contain only zeroes, while its one or two neighboring rows contain only -1's. Next, in any row containing only -1's, we first change the -1 in the odd-numbered columns (that is, the columns $1, 3, \ldots, n$) to zeroes, resulting in a row consisting of alternating 0 and -1 (since the -1's in the even-numbered columns have been changed two times), and we then easily obtain an entire row of zeroes. The effect of this on the next neighboring row is to create a new row of -1's, while the original row is clearly unchanged. In this way we finally obtain a zero table.

04.12. *There are* $2n$ *different numbers in a row. By one move we can interchange any two numbers or interchange any three numbers cyclically (choose* a, b, c *and place* a *instead of* b*,* b *instead of* c *and* c *instead of* a*). What is the minimal number of moves that is always sufficient to arrange the numbers in increasing order?*
Solution: If a number y occupies the place where x should be at the end, we draw an arrow $x \to y$. Clearly at the beginning all numbers are arranged in several cycles: Loops $\bullet \circlearrowleft$, binary cycles $\bullet \rightleftarrows \bullet$ and "long" cycles  (at least three numbers). Our aim is to obtain $2n$ loops.

Clearly each binary cycle can be rearranged into two loops by one move. If there is a long cycle with a fragment $\cdots \to a \to b \to c \to \cdots$, interchange a, b, c cyclically so that at least two loops, $a \circlearrowleft$, $b \circlearrowleft$, appear. By each of these moves, the number of loops increase by 2, so at most n moves are needed.

On the other hand, by checking all possible ways the two or three numbers can be distributed among disjoint cycles, it is easy to see that each of the allowed moves increases the number of disjoint cycles by at most two. Hence if the initial situation is one single loop, at least n moves are needed.

04.13. *The* 25 *member states of the European Union set up a committee with the following rules: (1) the committee should meet daily; (2) at each meeting, at least one member state should be represented; (3) at any two different meetings, a different set of member states should be represented; and (4) at the* n*'th meeting, for every* $k < n$*, the set of states represented should include at least one state that was represented at the* k*'th meeting. For how many days can the committee have its meetings?*
Answer: At most $2^{24} = 16777216$ days.
Solution: If one member is always represented, rules 2 and 4 will be fulfilled. There are 2^{24} different subsets of the remaining 24 members, so there can be at least 2^{24} meetings. Rule 3 forbids complementary sets at two different meetings, so the maximal number of meetings cannot exceed $\frac{1}{2} \cdot 2^{25} = 2^{24}$. So the maximal number of meetings for the committee is exactly $2^{24} = 16777216$.

04.14. *We say that a pile is a set of four or more nuts. Two persons play the following game. They start with one pile of* $n \geq 4$ *nuts. During a move a player takes one of the piles that they have and split it into two non-empty subsets (these sets are not necessarily piles, they can contain an arbitrary number of nuts). If the player cannot move, he loses. For which values of* n *does the first player have a winning strategy?*
Answer: The first player has a winning strategy when $n \equiv 0, 1, 2 \pmod 4$; otherwise the second player has a winning strategy.
Solution: Let $n = 4k + r$, where $0 \leq r \leq 3$. We will prove the above answer by induction on k; clearly it holds for $k = 1$. We are also going to need the following *useful fact*:

> If at some point there are exactly two piles with $4s + 1$ and $4t + 1$ nuts, $s + t \leq k$, then the second player to move from that point wins.

This holds vacuously when $k = 1$.

Now assume that we know the answer when the starting pile consists of at most $4k - 1$ nuts, and that the useful fact holds for $s + t \leq k$. We will prove the answer is correct for $4k$, $4k + 1$, $4k + 2$ and $4k + 3$, and that the useful fact holds for $s + t \leq k + 1$. For the sake of bookkeeping, we will refer to the first player as A and the second player as B.

If the pile consists of $4k$, $4k + 1$ or $4k + 2$ nuts, A simply makes one pile consisting of $4k - 1$ nuts, and another consisting of 1, 2 or 3 nuts, respectively. This makes A the second player in a game starting with $4k - 1 \equiv 3 \pmod 4$ nuts, so A wins.

Now assume the pile contains $4k + 3$ nuts. A can split the pile in two ways: Either as $(4p + 1, 4q + 2)$ or $(4p, 4q + 3)$. In the former case, if either p or q is 0, B wins by the above paragraph. Otherwise, B removes one nut from the $4q + 2$ pile, making B the second player in a game where we may apply the useful fact (since $p + q = k$), so B wins. If A splits the original pile as $(4p, 4q + 3)$, B removes one nut from the $4p$ pile, so the situation is two piles with $4(p - 1) + 3$ and $4q + 3$ nuts. Then B can use the winning strategy for the second player just described on each pile seperately, ultimately making B the winner.

It remains to prove the useful fact when $s + t = k + 1$. Due to symmetry, there are two possibilities for the first move: Assume the first player moves $(4s + 1, 4t + 1) \to (4s + 1, 4p, 4q + 1)$. The second player then splits the middle pile into $(4p - 1, 1)$, so the situation is $(4s + 1, 4q + 1, 4p - 1)$. Since the second player has a winning strategy both when the initial situtation is $(4s + 1, 4q + 1)$ and when it is $4p - 1$, he wins (this also holds when $p = 1$).

Now assume the first player makes the move $(4s + 1, 4t + 1) \to (4s + 1, 4p + 2, 4q + 3)$. If $p = 0$, the second player splits the third pile as $4q + 3 = (4q + 1) + 2$ and wins by the useful fact. If $p > 0$, the second player splits the second pile as $4p + 2 = (4p + 1) + 1$, and wins because he wins in each of the situations $(4s + 1, 4p + 1)$ and $4q + 3$.

04.15. *A circle is divided into* 13 *segments, numbered consecutively from* 1 *to* 13. *Five fleas called A, B, C, D and E are sitting in the segments* $1, 2, 3, 4$ *and* 5. *A flea is allowed to jump to an empty segment five positions away in either direction around the circle. Only one flea jumps at the same time, and two fleas cannot be in the same segment. After some jumps, the fleas are back in the segments* $1, 2, 3, 4, 5$, *but possibly in some other order than they started. Which orders are possible?*

Solution: Write the numbers from 1 to 13 in the order **1**, 6, 11, **3**, 8, 13, **5**, 10, **2**, 7, 12, **4**, 9. Then each time a flea jumps it moves between two adjacent numbers or between the first and the last number in this row. Since a flea can never move past another flea, the possible permutations are

1	3	5	2	4
A	C	E	B	D
D	A	C	E	B
B	D	A	C	E
E	B	D	A	C
C	E	B	D	A

or equivalently

1	2	3	4	5
A	B	C	D	E
D	E	A	B	C
B	C	D	E	A
E	A	B	C	D
C	D	E	A	B

that is, exactly the cyclic permutations of the original order.

04.16. *Through a point P exterior to a given circle pass a secant and a tangent to the circle. The secant intersects the circle at A and B, and the tangent touches the circle at C on the same side of the diameter thorugh P as A and B. The projection of C on the diameter is Q. Prove that QC bisects* $\angle AQB$.

Solution: Denoting the centre of the circle by O, we have $OQ \cdot OP = OA^2 = OB^2$. Hence $\triangle OAQ \sim \triangle OPA$ and $\triangle OBQ \sim \triangle OPB$. Since $\triangle AOB$ is isosceles, we have $\angle OAP + \angle OBP = 180^\circ$, and therefore

$$\begin{aligned}\angle AQP + \angle BQP &= \angle AOP + \angle OAQ + \angle BOP + \angle OBQ\\ &= \angle AOP + \angle OPA + \angle BOP + \angle OPB\\ &= 180^\circ - \angle OAP + 180^\circ - \angle OBP\\ &= 180^\circ.\end{aligned}$$

Thus QC, being perpendicular to QP, bisects $\angle AQB$.

04.17. *Consider a rectangle with side lengths 3 and 4, and pick an arbitrary inner point on each side. Let x, y, z and u denote the side lengths of the quadrilateral spanned by these points. Prove that $25 \le x^2 + y^2 + z^2 + u^2 \le 50$.*

Solution: Let a, b, c and d be the distances of the chosen points from the midpoints of the sides of the rectangle (with a and c on the sides of length 3). Then

$$\begin{aligned}x^2 + y^2 + z^2 + u^2 &= (\tfrac{3}{2} + a)^2 + (\tfrac{3}{2} - a)^2 + (\tfrac{3}{2} + c)^2 + (\tfrac{3}{2} - c)^2\\ &\qquad + (2 + b)^2 + (2 - b)^2 + (2 + d)^2 + (2 - d)^2\\ &= 4 \cdot (\tfrac{3}{2})^2 + 4 \cdot 2^2 + 2(a^2 + b^2 + c^2 + d^2)\\ &= 25 + 2(a^2 + b^2 + c^2 + d^2).\end{aligned}$$

Since $0 \le a^2, c^2 \le (3/2)^2$, $0 \le b^2, d^2 \le 2^2$, the desired inequalities follow.

04.18. *A ray emanating from the vertex A of the triangle ABC intersects the side BC at X and the circumcircle of ABC at Y. Prove that $\frac{1}{AX} + \frac{1}{XY} \ge \frac{4}{BC}$.*

Solution: From the GM-HM inequality we have

$$\frac{1}{AX} + \frac{1}{XY} \ge \frac{2}{\sqrt{AX \cdot XY}}. \tag{04.16}$$

As BC and AY are chords intersecting at X we have $AX \cdot XY = BX \cdot XC$. Therefore (04.16) transforms into

$$\frac{1}{AX} + \frac{1}{XY} \ge \frac{2}{\sqrt{BX \cdot XC}}. \tag{04.17}$$

We also have

$$\sqrt{BX \cdot XC} \le \frac{BX + XC}{2} = \frac{BC}{2},$$

so from (04.17) the result follows.

04.19. *D is the midpoint of the side BC of the given triangle ABC. M is a point on the side BC such that $\angle BAM = \angle DAC$. L is the second intersection point of the circumcircle of the triangle CAM with the side AB. K is the second intersection point of the circumcircle of the triangle BAM with the side AC. Prove that $KL \parallel BC$.*

Solution: It is sufficient to prove that $CK : LB = AC : AB$.

The triangles ABC and MKC are similar beacuse they have common angle C and $\angle CMK = 180^\circ - \angle BMK = \angle KAB$ (the latter equality is due to the observation that $\angle BMK$ and $\angle KAB$ are the opposite angles in the insecribed quadrilateral $AKMB$).

By analogous reasoning the triangles ABC and MBL are similar. Therefore the triangles MKC and MBL are also similar and we have

$$\frac{CK}{LB} = \frac{KM}{BM} = \frac{\frac{AM\sin KAM}{\sin AKM}}{\frac{AM\sin MAB}{\sin MBA}} = \frac{\sin KAM}{\sin MAB} = \frac{\sin DAB}{\sin DAC} = \frac{\frac{BD\sin BDA}{AB}}{\frac{CD\sin CDA}{AC}} = \frac{AC}{AB}.$$

The second equality is due to the sinus theorem for triangles AKM and ABM; the third is due to the equality $\angle AKM = 180^\circ - \angle MBA$ in the inscribed quadrilateral $AKMB$; the fourth is due to the definition of the point M; and the fifth is due to the sinus theorem for triangles ACD and ABD.

04.20. *Three circular arcs w_1, w_2, w_3 with common endpoints A and B are on the same side of the line AB; w_2 lies between w_1 and w_3. Two rays emanating from B intersect these arcs at M_1, M_2, M_3 and K_1, K_2, K_3, respectively. Prove that $\frac{M_1M_2}{M_2M_3} = \frac{K_1K_2}{K_2K_3}$.*

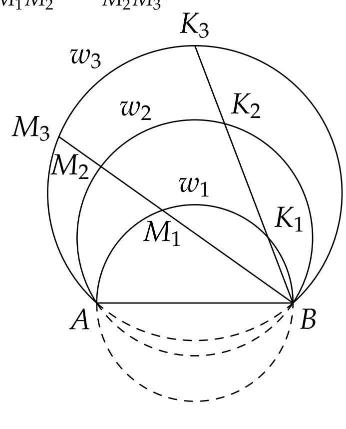
Solution: From inscribed angles we have $\angle AK_1B = \angle AM_1B$ and $\angle AK_2B = \angle AM_2B$. From this it follows that $\triangle AK_1K_2 \sim \triangle AM_1M_2$, so

$$\frac{K_1K_2}{M_1M_2} = \frac{AK_2}{AM_2}.$$

Similarly $\triangle AK_2K_3 \sim \triangle AM_2M_3$, so

$$\frac{K_2K_3}{M_2M_3} = \frac{AK_2}{AM_2}.$$

From these equations we get $\frac{K_1K_2}{M_1M_2} = \frac{K_2K_3}{M_2M_3}$, from which the desired property follows.



Baltic Way 2005

05.1. *Let a_0 be a positive integer. Define the sequence a_n, $n \geq 0$, as follows: If*

$$a_n = \sum_{i=0}^{j} c_i 10^i$$

where c_i are integers with $0 \leq c_i \leq 9$, then

$$a_{n+1} = c_0^{2005} + c_1^{2005} + \cdots + c_j^{2005}.$$

Is it possible to choose a_0 so that all the terms in the sequence are distinct?
Answer: No, the sequence must contain two equal terms.
Solution: It is clear that there exists a smallest positive integer k such that

$$10^k > (k+1) \cdot 9^{2005}.$$

We will show that there exists a positive integer N such that a_n consists of less than $k+1$ decimal digits for all $n \geq N$. Let a_i be a positive integer which consists of exactly $j+1$ digits, that is,

$$10^j \leq a_i < 10^{j+1}.$$

We need to prove two statements:

- a_{i+1} has less than $k+1$ digits if $j < k$; and
- $a_i > a_{i+1}$ if $j \geq k$.

To prove the first statement, notice that

$$a_{i+1} \leq (j+1) \cdot 9^{2005} < (k+1) \cdot 9^{2005} < 10^k$$

and hence a_{i+1} consists of less than $k+1$ digits. To prove the second statement, notice that a_i consists of $j+1$ digits, none of which exceeds 9. Hence $a_{i+1} \leq (j+1) \cdot 9^{2005}$ and because $j \geq k$, we get $a_i \geq 10^j > (j+1) \cdot 9^{2005} \geq a_{i+1}$, which proves the second statement. It is now easy to derive the result from this statement. Assume that a_0 consists of $k+1$ or more digits (otherwise we are done, because then it follows inductively that all terms of the sequence consist of less than $k+1$ digits, by the first statement). Then the sequence starts with a strictly decreasing segment $a_0 > a_1 > a_2 > \cdots$ by the second statement, so for some index N the number a_N has less than $k+1$ digits. Then, by the first statement, each number a_n with $n \geq N$ consists of at most k digits. By the Pigeonhole Principle, there are two different indices $n, m \geq N$ such that $a_n = a_m$.

05.2. *Let α, β and γ be three angles with $0 \leq \alpha, \beta, \gamma < 90^\circ$ and $\sin\alpha + \sin\beta + \sin\gamma = 1$. Show that*

$$\tan^2\alpha + \tan^2\beta + \tan^2\gamma \geq \frac{3}{8}.$$

Solution: Since $\tan^2 x = 1/\cos^2 x - 1$, the inequality to be proved is equivalent to

$$\frac{1}{\cos^2\alpha} + \frac{1}{\cos^2\beta} + \frac{1}{\cos^2\gamma} \geq \frac{27}{8}.$$

The AM-HM inequality implies

$$\begin{aligned}\frac{3}{\frac{1}{\cos^2\alpha}+\frac{1}{\cos^2\beta}+\frac{1}{\cos^2\gamma}} &\leq \frac{\cos^2\alpha+\cos^2\beta+\cos^2\gamma}{3}\\ &= \frac{3-(\sin^2\alpha+\sin^2\beta+\sin^2\gamma)}{3}\\ &\leq 1-\Big(\frac{\sin\alpha+\sin\beta+\sin\gamma}{3}\Big)^2\\ &= \frac{8}{9}\end{aligned}$$

and the result follows.

05.3. *Consider the sequence a_k defined by $a_1 = 1$, $a_2 = \frac{1}{2}$,*

$$a_{k+2} = a_k + \frac{1}{2}a_{k+1} + \frac{1}{4a_ka_{k+1}} \quad \textit{for } k \geq 1.$$

Prove that

$$\frac{1}{a_1a_3}+\frac{1}{a_2a_4}+\frac{1}{a_3a_5}+\cdots+\frac{1}{a_{98}a_{100}} < 4.$$

Solution: Note that

$$\frac{1}{a_ka_{k+2}} < \frac{2}{a_ka_{k+1}} - \frac{2}{a_{k+1}a_{k+2}},$$

because this inequality is equivalent to the inequality

$$a_{k+2} > a_k + \frac{1}{2}a_{k+1},$$

which is evident for the given sequence. Now we have

$$\begin{aligned}&\frac{1}{a_1a_3}+\frac{1}{a_2a_4}+\frac{1}{a_3a_5}+\cdots+\frac{1}{a_{98}a_{100}}\\ &\quad< \frac{2}{a_1a_2}-\frac{2}{a_2a_3}+\frac{2}{a_2a_3}-\frac{2}{a_3a_4}+\cdots\\ &\quad< \frac{2}{a_1a_2} = 4.\end{aligned}$$

05.4. *Find three different polynomials $P(x)$ with real coefficients such that $P(x^2+1) = P(x)^2+1$ for all real x.*

Answer: For example, $P(x) = x$, $P(x) = x^2+1$ and $P(x) = x^4+2x^2+2$.

Solution: Let $Q(x) = x^2+1$. Then the equation that P must satisfy can be written $P(Q(x)) = Q(P(x))$, and it is clear that this will be satisfied for $P(x) = x$, $P(x) = Q(x)$ and $P(x) = Q(Q(x))$.

Solution 2: For all reals x we have $P(x)^2+1 = P(x^2+1) = P(-x)^2+1$ and consequently, $(P(x)+P(-x))(P(x)-P(-x)) = 0$. Now one of the three cases holds:

(a) If both $P(x)+P(-x)$ and $P(x)-P(-x)$ are not identically 0, then they are non-constant polynomials and have a finite numbers of roots, so this case cannot hold.

(b) If $P(x)+P(-x)$ is identically 0 then obviously, $P(0)=0$. Consider the infinite sequence of integers $a_0=0$ and $a_{n+1}=a_n^2+1$. By induction it is easy to see that $P(a_n)=a_n$ for all non-negative integers n. Also, $Q(x)=x$ has that property, so $P(x)-Q(x)$ is a polynomial with infinitely many roots, whence $P(x)=x$.

(c) If $P(x)-P(-x)$ is identically 0 then

$$P(x)=x^{2n}+b_{n-1}x^{2n-2}+\cdots+b_1x^2+b_0,$$

for some integer n since $P(x)$ is even and it is easy to see that the coefficient of x^{2n} must be 1. Putting $n=1$ and $n=2$ yield the solutions $P(x)=x^2+1$ and $P(x)=x^4+2x^2+2$.

Remark: For $n=3$ there is no solution, whereas for $n=4$ there is the unique solution $P(x)=x^8+6x^6+8x^4+8x^2+5$.

05.5. *Let a,b,c be positive real numbers with $abc=1$. Prove that*

$$\frac{a}{a^2+2}+\frac{b}{b^2+2}+\frac{c}{c^2+2}\le 1.$$

Solution: For any positive real x we have $x^2+1\ge 2x$. Hence

$$\begin{aligned}\frac{a}{a^2+2}+\frac{b}{b^2+2}+\frac{c}{c^2+2}&\le\frac{a}{2a+1}+\frac{b}{2b+1}+\frac{c}{2c+1}\\&=\frac{1}{2+1/a}+\frac{1}{2+1/b}+\frac{1}{2+1/c}=:R.\end{aligned}$$

$R\le 1$ is equivalent to

$$\left(2+\tfrac{1}{b}\right)\left(2+\tfrac{1}{c}\right)+\left(2+\tfrac{1}{a}\right)\left(2+\tfrac{1}{c}\right)+\left(2+\tfrac{1}{a}\right)\left(2+\tfrac{1}{b}\right)\le\left(2+\tfrac{1}{a}\right)\left(2+\tfrac{1}{b}\right)\left(2+\tfrac{1}{c}\right)$$

and to $4\le\frac{1}{ab}+\frac{1}{ac}+\frac{1}{bc}+\frac{1}{abc}$. By $abc=1$ and by the AM-GM inequality

$$\frac{1}{ab}+\frac{1}{ac}+\frac{1}{bc}\ge 3\sqrt[3]{\left(\frac{1}{abc}\right)^2}=3$$

the last inequality follows. Equality appears exactly when $a=b=c=1$.

05.6. *Let K and N be positive integers with $1\le K\le N$. A deck of N different playing cards is shuffled by repeating the operation of reversing the order of the K topmost cards and moving these to the bottom of the deck. Prove that the deck will be back in its initial order after a number of operations not greater than $4\cdot N^2/K^2$.*

Solution: Let $N=q\cdot K+r$, $0\le r<K$, and let us number the cards $1,2,\ldots,N$, starting from the one at the bottom of the deck. First we find out how the cards $1,2,\ldots K$ are moving in the deck.

If $i\le r$ then the card i is moving along the cycle

$$\begin{aligned}&i\to K+i\to 2K+i\to\cdots\to qK+i\to(r+1-i)\to\\&\qquad K+(r+1-i)\to\cdots\to qK+(r+1-i),\end{aligned}$$

because $N-K<qK+i\le N$ and $N-K<qK+(r+1-i)\le N$. The length of this cycle is $2q+2$. In the special case of $i=r+i-1$, it actually consists of two smaller cycles of length $q+1$.

If $r < i \leq K$ then the card i is moving along the cycle

$$i \to K+i \to 2K+i \to \cdots \to (q-1)K+i \to \\ K+r+1-i \to K+(K+r+1-i) \to \\ 2K+(K+r+1-i) \to \cdots \to (q-1)K+(K+r+1-i),$$

because $N-K < (q-1)K+i \leq N$ and $N-K < (q-1)K+(K+r+1-i) \leq N$. The length of this cycle is $2q$. In the special case of $i = K+r+1-i$, it actually consists of two smaller cycles of length q.

Since these cycles cover all the numbers $1, \ldots, N$, we can say that every card returns to its initial position after either $2q+2$ or $2q$ operations. Therefore, all the cards are simultaneously at their initial position after at most $\operatorname{lcm}(2q+2, 2q) = 2\operatorname{lcm}(q+1, q) = 2q(q+1)$ operations. Finally,

$$2q(q+1) \leq (2q)^2 = 4q^2 \leq 4\left(\tfrac{N}{K}\right)^2,$$

which concludes the proof.

05.7. *A rectangular array has n rows and six columns, where $n > 2$. In each cell there is written either 0 or 1. All rows in the array are different from each other. For each pair of rows $(x_1, x_2, \ldots, x_6)$ and $(y_1, y_2, \ldots, y_6)$, the row $(x_1y_1, x_2y_2, \ldots, x_6y_6)$ can also be found in the array. Prove that there is a column in which at least half of the entries are zeroes.*

Solution: Clearly there must be rows with some zeroes. Consider the case when there is a row with just one zero; we can assume it is $(0,1,1,1,1,1)$. Then for each row $(1, x_2, x_3, x_4, x_5, x_6)$ there is also a row $(0, x_2, x_3, x_4, x_5, x_6)$; the conclusion follows. Consider the case when there is a row with just two zeroes; we can assume it is $(0,0,1,1,1,1)$. Let n_{ij} be the number of rows with first two elements i, j. As in the first case $n_{00} \geq n_{11}$. Let $n_{01} \geq n_{10}$; the other subcase is analogous. Now there are $n_{00} + n_{01}$ zeroes in the first column and $n_{10} + n_{11}$ ones in the first column; the conclusion follows. Consider now the case when each row contains at least three zeroes (except $(1,1,1,1,1,1)$, if such a row exists). Let us prove that it is impossible that each such row contains exactly three zeroes. Assume the opposite. As $n > 2$ there are at least two rows with zeroes; they are different, so their product contains at least four zeroes, a contradiction. So there are more then $3(n-1)$ zeroes in the array; so in some column there are more than $(n-1)/2$ zeroes; so there are at least $n/2$ zeroes.

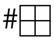
05.8. *Consider a grid of 25×25 unit squares. Draw with a red pen contours of squares of any size on the grid. What is the minimal number of squares we must draw in order to colour all the lines of the grid?*

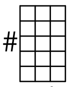
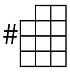
Answer: 48 squares.

Solution: Consider a diagonal of the square grid. For any grid vertex A on this diagonal denote by C the farthest endpoint of this diagonal. Let the square with the diagonal AC be red. Thus, we have defined the set of 48 red squares (24 for each diagonal). It is clear that if we draw all these squares, all the lines in the grid will turn red.

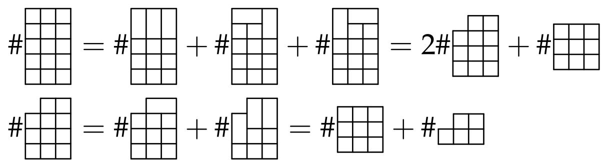
In order to show that 48 is the minimum, consider all grid segments of length 1 that have exactly one endpoint on the border of the grid. Every horizontal and every vertical line that cuts the grid into two parts determines two such segments. So we have $4 \cdot 24 = 96$ segments. It is evident that every red square can contain at most two of these segments.

05.9. *A rectangle is divided into 200×3 unit squares. Prove that the number of ways of splitting this rectangle into rectangles of size 1×2 is divisible by 3.*

Solution: Let us denote the number of ways to split some figure into dominos by a small picture of this figure with a sign #. For example, # = 2.

Let N_n = # (n rows) and γ_n = # ($n-2$ full rows and one row with two cells). We are going to find a recurrence relation for the numbers N_n.

Observe that



We can generalize our observations by writing the equalities

$$\begin{aligned} N_n &= 2\gamma_n + N_{n-2}, \\ 2\gamma_{n-2} &= N_{n-2} - N_{n-4}, \\ 2\gamma_n &= 2\gamma_{n-2} + 2N_{n-2}. \end{aligned}$$

If we sum up these equalities we obtain the desired recurrence

$$N_n = 4N_{n-2} - N_{n-4}.$$

It is easy to find that $N_2 = 3$, $N_4 = 11$. Now by the recurrence relation it is trivial to check that $N_{6k+2} \equiv 0 \pmod 3$.

05.10. *Let $m = 30030 = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13$ and let M be the set of its positive divisors which have exactly two prime factors. Determine the minimal integer n with the following property: for any choice of n numbers from M, there exist three numbers a, b, c among them satisfying $a \cdot b \cdot c = m$.*

Answer: $n = 11$.

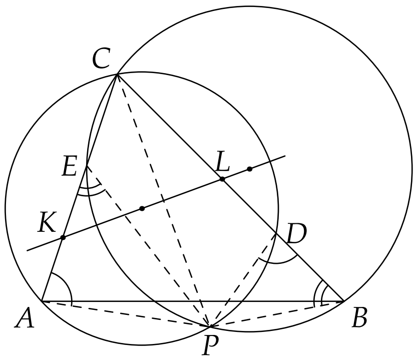
Solution: Taking the 10 divisors without the prime 13 shows that $n \geq 11$. Consider the following partition of the 15 divisors into five groups of three each with the property that the product of the numbers in every group equals m.

$$\{2\cdot 3, 5\cdot 13, 7\cdot 11\}, \quad \{2\cdot 5, 3\cdot 7, 11\cdot 13\}, \quad \{2\cdot 7, 3\cdot 13, 5\cdot 11\},$$
$$\{2\cdot 11, 3\cdot 5, 7\cdot 13\}, \quad \{2\cdot 13, 3\cdot 11, 5\cdot 7\}.$$

If $n = 11$, there is a group from which we take all three numbers, that is, their product equals m.

05.11. *Let the points D and E lie on the sides BC and AC, respectively, of the triangle ABC, satisfying $BD = AE$. The line joining the circumcentres of the triangles ADC and BEC meets the lines AC and BC at K and L, respectively. Prove that $KC = LC$.*

Solution: Assume that the circumcircles of triangles ADC and BEC meet at C and P. The problem is to show that the line KL makes equal angles with the lines AC and BC. Since the line joining the circumcentres of triangles ADC and BEC is perpendicular to the line CP, it suffices to show that CP is the angle-bisector of $\angle ACB$.



Since the points A, P, D, C are concyclic, we obtain $\angle EAP = \angle BDP$. Analogously, we have $\angle AEP = \angle DBP$. These two equalities together with $AE = BD$ imply that triangles APE and DPB are congruent. This means that the distance from P to AC is equal to the distance from P to BC, and thus CP is the angle-bisector of $\angle ACB$, as desired.

05.12. *Let $ABCD$ be a convex quadrilateral such that $BC = AD$. Let M and N be the midpoints of AB and CD, respectively. The lines AD and BC meet the line MN at P and Q, respectively. Prove that $CQ = DP$.*

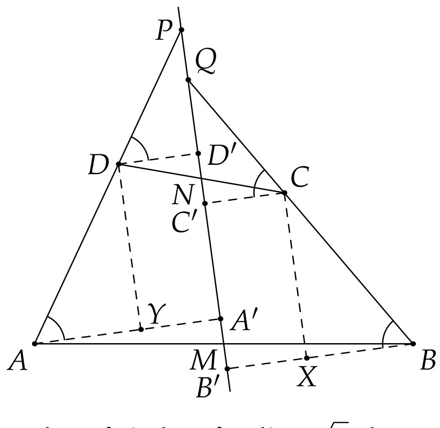
Solution: Let A', B', C', D' be the feet of the perpendiculars from A, B, C, D, respectively, onto the line MN. Then

$$AA' = BB' \qquad \text{and} \qquad CC' = DD'.$$

Denote by X, Y the feet of the perpendiculars from C, D onto the lines BB', AA', respectively. We infer from the above equalities that $AY = BX$. Since also $BC = AD$, the right-angled triangles BXC and AYD are congruent. This shows that

$$\angle C'CQ = \angle B'BQ = \angle A'AP = \angle D'DP.$$

Therefore, since $CC' = DD'$, the triangles $CC'Q$ and $DD'P$ are congruent. Thus $CQ = DP$.



05.13. *What is the smallest number of circles of radius $\sqrt{2}$ that are needed to cover a rectangle*

(a) *of size* 6×3?

(b) *of size* 5×3?

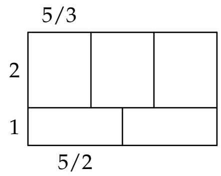
Answer: (a) Six circles, (b) five circles.

Solution: (a) Consider the four corners and the two midpoints of the sides of length 6. The distance between any two of these six points is 3 or more, so one circle cannot cover two of these points, and at least six circles are needed.

On the other hand one circle will cover a 2×2 square, and it is easy to see that six such squares can cover the rectangle.

(b) Consider the four corners and the centre of the rectangle. The minimum distance between any two of these points is the distance between the centre and one of the corners, which is $\sqrt{34}/2$. This is greater than the diameter of the circle ($\sqrt{34/4} > \sqrt{32/4}$), so one circle cannot cover two of these points, and at least five circles are needed.

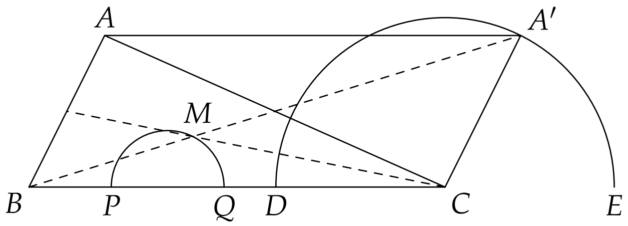
Partition the rectangle into three rectangles of size $5/3 \times 2$ and two rectangles of size $5/2 \times 1$ as shown on the right. It is easy to check that each has a diagonal of length less than $2\sqrt{2}$, so five circles can cover the five small rectangles and hence the 5×3 rectangle.



05.14. *Let the medians of the triangle ABC meet at M. Let D and E be different points on the line BC such that $DC = CE = AB$, and let P and Q be points on the segments BD and BE, respectively, such that $2BP = PD$ and $2BQ = QE$. Determine $\angle PMQ$.*

Answer: $\angle PMQ = 90°$.

Solution: Draw the parallelogram $ABCA'$, with $AA' \parallel BC$. Then M lies on BA', and $BM = \frac{1}{3}BA'$. So M is on the homothetic image (centre B, dilation $1/3$) of the circle with centre C and radius AB, which meets BC at D and E. The image meets BC at P and Q. So $\angle PMQ = 90°$.

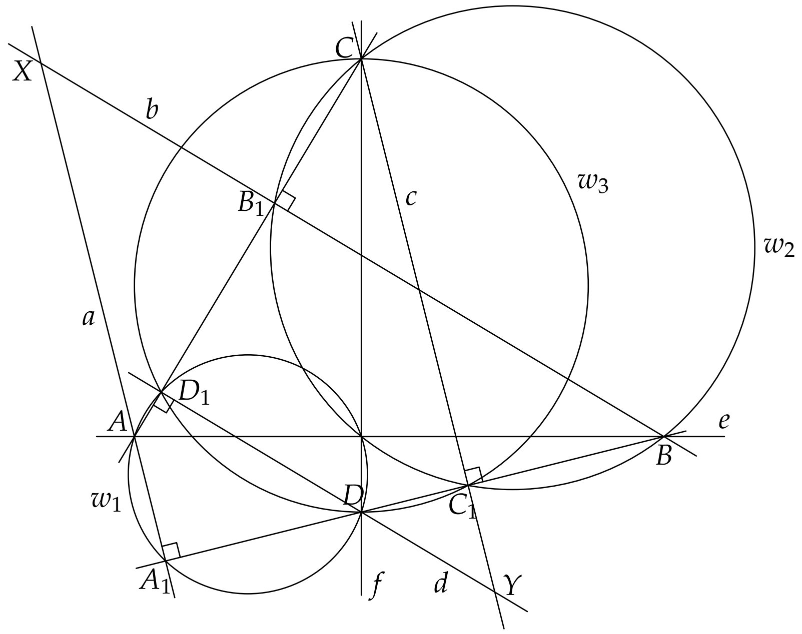


05.15. *Let the lines e and f be perpendicular and intersect each other at O. Let A and B lie on e and C and D lie on f, such that all the five points A, B, C, D and O are distinct. Let the lines b and d pass through B and D respectively, perpendicularly to AC; let the lines a and c pass through A and C respectively, perpendicularly to BD. Let a and b intersect at X and c and d intersect at Y. Prove that XY passes through O.*

Solution: Let A_1 be the intersection of a with BD, B_1 the intersection of b with AC, C_1 the intersection of c with BD and D_1 the intersection of d with AC. It follows easily by the given right angles that the following three sets each are concyclic:

- A, A_1, D, D_1, O lie on a circle w_1 with diameter AD.
- B, B_1, C, C_1, O lie on a circle w_2 with diameter BC.
- C, C_1, D, D_1 lie on a circle w_3 with diameter DC.

We see that O lies on the radical axis of w_1 and w_2. Also, Y lies on the radical axis of w_1 and w_3, and on the radical axis of w_2 and w_3, so Y is the radical centre of w_1, w_2 and w_3, so it lies on the radical axis of w_1 and w_2. Analogously we prove that X lies on the radical axis of w_1 and w_2.



05.16. *Let p be a prime number and let n be a positive integer. Let q be a positive divisor of $(n+1)^p - n^p$. Show that $q-1$ is divisible by p.*
Solution: It is sufficient to show the statement for q prime. We need to prove that

$$(n+1)^p \equiv n^p \pmod q \implies q \equiv 1 \pmod p.$$

It is obvious that $\gcd(n,q) = \gcd(n+1,q) = 1$ (as n and $n+1$ cannot be divisible by q simultaneously). Hence there exists a positive integer m such that $mn \equiv 1 \pmod q$. In fact, m is just the multiplicative inverse of $n \pmod q$. Take $s = m(n+1)$. It is easy to see that

$$s^p \equiv 1 \pmod q.$$

Let t be the smallest positive integer which satisfies $s^t \equiv 1 \pmod q$ (t is the order of $s \pmod q$). One can easily prove that t divides p. Indeed, write $p = at + b$ where $0 \le b < t$. Then

$$1 \equiv s^p \equiv s^{at+b} \equiv \left(s^t\right)^a \cdot s^b \equiv s^b \pmod q.$$

By the definition of t, we must have $b = 0$. Hence t divides p. This means that $t = 1$ or $t = p$. However, $t = 1$ is easily seen to give a contradiction since then we would have

$$m(n+1) \equiv 1 \pmod q \qquad \text{or} \qquad n+1 \equiv n \pmod q.$$

Therefore $t = p$, and p is the order of $s \pmod q$. By Fermat's little theorem,

$$s^{q-1} \equiv 1 \pmod q.$$

Since p is the order of $s \pmod q$, we have that p divides $q-1$, and we are done.

05.17. *A sequence (x_n), $n \ge 0$, is defined as follows: $x_0 = a$, $x_1 = 2$ and $x_n = 2x_{n-1}x_{n-2} - x_{n-1} - x_{n-2} + 1$ for $n > 1$. Find all integers a such that $2x_{3n} - 1$ is a perfect square for all $n \ge 1$.*
Answer: $a = \frac{(2m-1)^2+1}{2}$ where m is an arbitrary positive integer.

Solution: Let $y_n = 2x_n - 1$. Then

$$\begin{aligned} y_n &= 2(2x_{n-1}x_{n-2} - x_{n-1} - x_{n-2} + 1) - 1 \\ &= 4x_{n-1}x_{n-2} - 2x_{n-1} - 2x_{n-2} + 1 \\ &= (2x_{n-1} - 1)(2x_{n-2} - 1) = y_{n-1}y_{n-2} \end{aligned}$$

when $n > 1$. Notice that $y_{n+3} = y_{n+2}y_{n+1} = y_{n+1}^2 y_n$. We see that y_{n+3} is a perfect square if and only if y_n is a perfect square. Hence y_{3n} is a perfect square for all $n \geq 1$ exactly when y_0 is a perfect square. Since $y_0 = 2a - 1$, the result is obtained when $a = \frac{(2m-1)^2+1}{2}$ for all positive integers m.

05.18. *Let x and y be positive integers and assume that $z = 4xy/(x+y)$ is an odd integer. Prove that at least one divisor of z can be expressed in the form $4n - 1$ where n is a positive integer.*

Solution: Let $x = 2^s x_1$ and $y = 2^t y_1$ where x_1 and y_1 are odd integers. Without loss of generality we can assume that $s \geq t$. We have

$$z = \frac{2^{s+t+2}x_1y_1}{2^t(2^{s-t}x_1 + y_1)} = \frac{2^{s+2}x_1y_1}{2^{s-t}x_1 + y_1}.$$

If $s \neq t$, then the denominator is odd and therefore z is even. So we have $s = t$ and $z = 2^{s+2}x_1y_1/(x_1 + y_1)$. Let $x_1 = dx_2$, $y_1 = dy_2$ with $\gcd(x_2, y_2) = 1$. So $z = 2^{s+2}dx_2y_2/(x_2 + y_2)$. As z is odd, it must be that $x_2 + y_2$ is divisible by $2^{s+2} \geq 4$, so $x_2 + y_2$ is divisible by 4. As x_2 and y_2 are odd integers, one of them, say x_2 is congruent to 3 modulo 4. But $\gcd(x_2, x_2 + y_2) = 1$, so x_2 is a divisor of z.

05.19. *Is it possible to find 2005 different positive square numbers such that their sum is also a square number?*

Answer: Yes, it is possible.

Solution: Start with a simple Pythagorian identity such as $3^2 + 4^2 = 5^2$. Multiply it by 5^2

$$3^2 \cdot 5^2 + 4^2 \cdot 5^2 = 5^2 \cdot 5^2$$

and insert the identity for the first

$$3^2 \cdot (3^2 + 4^2) + 4^2 \cdot 5^2 = 5^2 \cdot 5^2$$

which gives

$$3^2 \cdot 3^2 + 3^2 \cdot 4^2 + 4^2 \cdot 5^2 = 5^2 \cdot 5^2.$$

Multiply again by 5^2

$$3^2 \cdot 3^2 \cdot 5^2 + 3^2 \cdot 4^2 \cdot 5^2 + 4^2 \cdot 5^2 \cdot 5^2 = 5^2 \cdot 5^2 \cdot 5^2$$

and split the first term

$$3^2 \cdot 3^2 \cdot (3^2 + 4^2) + 3^2 \cdot 4^2 \cdot 5^2 + 4^2 \cdot 5^2 \cdot 5^2 = 5^2 \cdot 5^2 \cdot 5^2$$

that is

$$3^2 \cdot 3^2 \cdot 3^2 + 3^2 \cdot 3^2 \cdot 4^2 + 3^2 \cdot 4^2 \cdot 5^2 + 4^2 \cdot 5^2 \cdot 5^2 = 5^2 \cdot 5^2 \cdot 5^2.$$

This (multiplying by 5^2 and splitting the first term) can be repeated as often as needed, each time increasing the number of terms by one.

Clearly, each term is a square number and the terms are strictly increasing from left to right.

05.20. *Find all positive integers* $n = p_1 p_2 \cdots p_k$ *which divide* $(p_1+1)(p_2+1)\cdots(p_k+1)$, *where* $p_1 p_2 \cdots p_k$ *is the factorization of* n *into prime factors (not necessarily distinct).*
Answer: All numbers $2^r 3^s$ where r and s are non-negative integers and $s \le r \le 2s$.
Solution: Let $m = (p_1+1)(p_2+1)\cdots(p_k+1)$. We may assume that p_k is the largest prime factor. If $p_k > 3$ then p_k cannot divide m, because if p_k divides m it is a prime factor of $p_i + 1$ for some i, but if $p_i = 2$ then $p_i + 1 < p_k$, and otherwise $p_i + 1$ is an even number with factors 2 and $\frac{1}{2}(p_i + 1)$ which are both strictly smaller than p_k. Thus the only primes that can divide n are 2 and 3, so we can write $n = 2^r 3^s$. Then $m = 3^r 4^s = 2^{2s} 3^r$ which is divisible by n if and only if $s \le r \le 2s$.

Baltic Way 2006

06.1. *For a sequence $a_1, a_2, a_3, \dots$ of real numbers it is known that*

$$a_n = a_{n-1} + a_{n+2} \qquad \text{for } n = 2, 3, 4, \dots.$$

What is the largest number of its consecutive elements that can all be positive?
Answer: 5.
Solution: The initial segment of the sequence could be 1; 2; 3; 1; 1; −2; 0. Clearly it is enough to consider only initial segments. For each sequence the first 6 elements are a_1; a_2; a_3; $a_2 - a_1$; $a_3 - a_2$; $a_2 - a_1 - a_3$. As we see, $a_1 + a_5 + a_6 = a_1 + (a_3 - a_2) + (a_2 - a_1 - a_3) = 0$. So all the elements a_1, a_5, a_6 can not be positive simultaneously.

06.2. *Suppose that the real numbers $a_i \in [-2, 17]$, $i = 1, 2, \dots, 59$, satisfy $a_1 + a_2 + \cdots + a_{59} = 0$. Prove that*

$$a_1^2 + a_2^2 + \cdots + a_{59}^2 \le 2006.$$

Solution: For convenience denote $m = -2$ and $M = 17$. Then

$$\Big(a_i - \frac{m+M}{2}\Big)^2 \le \Big(\frac{M-m}{2}\Big)^2,$$

because $m \le a_i \le M$. So we have

$$\sum_{i=1}^{59}\Big(a_i - \frac{m+M}{2}\Big)^2 = \sum_i a_i^2 + 59 \cdot \Big(\frac{m+M}{2}\Big)^2 - (m+M)\sum_i a_i \le 59 \cdot \Big(\frac{M-m}{2}\Big)^2,$$

and thus

$$\sum_i a_i^2 \le 59 \cdot \bigg(\Big(\frac{M-m}{2}\Big)^2 - \Big(\frac{m+M}{2}\Big)^2\bigg) = -59 \cdot m \cdot M = 2006.$$

06.3. *Prove that for every polynomial $P(x)$ with real coefficients there exist a positive integer m and polynomials $P_1(x), P_2(x), \dots, P_m(x)$ with real coefficients such that*

$$P(x) = \big(P_1(x)\big)^3 + \big(P_2(x)\big)^3 + \cdots + \big(P_m(x)\big)^3.$$

Solution: We will prove by induction on the degree of $P(x)$ that all polynomials can be represented as a sum of cubes. This is clear for constant polynomials. Now we proceed to the inductive step. It is sufficient to show that if $P(x)$ is a polynomial of degree n, then there exist polynomials $Q_1(x)$, $Q_2(x)$, …, $Q_r(x)$ such that the polynomial

$$P(x) - (Q_1(x))^3 - (Q_2(x))^3 - \cdots - (Q_r(x))^3$$

has degree at most $n - 1$. Assume that the coefficient of x^n in $P(x)$ is equal to c. We consider three cases: If $n = 3k$, we put $r = 1$, $Q_1(x) = \sqrt[3]{c}x^k$; if $n = 3k+1$ we put $r = 3$,

$$Q_1(x) = \sqrt[3]{\frac{c}{6}}x^k(x-1), \qquad Q_2(x) = \sqrt[3]{\frac{c}{6}}x^k(x+1), \qquad Q_3(x) = -\sqrt[3]{\frac{c}{3}}x^{k+1};$$

and if $n = 3k+2$ we put $r = 2$ and

$$Q_1(x) = \sqrt[3]{\frac{c}{3}}x^k(x+1), \qquad Q_2(x) = -\sqrt[3]{\frac{c}{3}}x^{k+1}.$$

This completes the induction.

06.4. *Let a, b, c, d, e, f be non-negative real numbers satisfying $a+b+c+d+e+f=6$. Find the maximal possible value of*

$$abc+bcd+cde+def+efa+fab$$

and determine all 6-tuples (a,b,c,d,e,f) for which this maximal value is achieved.
Answer: 8.
Solution: If we set $a=b=c=2$, $d=e=f=0$, then the given expression is equal to 8. We will show that this is the maximal value. Applying the inequality between arithmetic and geometric mean we obtain

$$\begin{aligned} 8 &= \Big(\frac{(a+d)+(b+e)+(c+f)}{3}\Big)^3 \geq (a+d)(b+e)(c+f) \\ &= (abc+bcd+cde+def+efa+fab)+(ace+bdf), \end{aligned}$$

so we see that $abc+bcd+cde+def+efa+fab \leq 8$ and the maximal value 8 is achieved when $a+d=b+e=c+f$ (and then the common value is 2 because $a+b+c+d+e+f=6$) and $ace=bdf=0$, which can be written as $(a,b,c,d,e,f)=(a,b,c,2-a,2-b,2-c)$ with $ac(2-b)=b(2-a)(2-c)=0$. From this it follows that (a,b,c) must have one of the forms $(0,0,t)$, $(0,t,2)$, $(t,2,2)$, $(2,2,t)$, $(2,t,0)$ or $(t,0,0)$. Therefore the maximum is achieved for the 6-tuples $(a,b,c,d,e,f)=(0,0,t,2,2,2-t)$, where $0\leq t\leq 2$, and its cyclic permutations.

06.5. *An occasionally unreliable professor has devoted his last book to a certain binary operation $*$. When this operation is applied to any two integers, the result is again an integer. The operation is known to satisfy the following axioms:*

(a) *$x*(x*y)=y$ for all $x,y\in\mathbb{Z}$;*

(b) *$(x*y)*y=x$ for all $x,y\in\mathbb{Z}$.*

The professor claims in his book that

(C1) *the operation $*$ is commutative: $x*y=y*x$ for all $x,y\in\mathbb{Z}$.*

(C2) *the operation $*$ is associative: $(x*y)*z=x*(y*z)$ for all $x,y,z\in\mathbb{Z}$.*

Which of these claims follow from the stated axioms?
Answer: (C1) is true; (C2) is false.
Solution: Write (x,y,z) for $x*y=z$. So the axioms can be formulated as

$$(x,y,z) \implies (x,z,y) \tag{06.18}$$
$$(x,y,z) \implies (z,y,x). \tag{06.19}$$

(C1) is proved by the sequence $(x,y,z) \overset{(06.19)}{\implies} (z,y,x) \overset{(06.18)}{\implies} (z,x,y) \overset{(06.19)}{\implies} (y,x,z)$.
A counterexample for (C2) is the operation $x*y=-(x+y)$.

06.6. *Determine the maximal size of a set of positive integers with the following properties:*

(1) *The integers consist of digits from the set $\{1,2,3,4,5,6\}$.*

(2) *No digit occurs more than once in the same integer.*

(3) *The digits in each integer are in increasing order.*

(4) *Any two integers have at least one digit in common (possibly at different positions).*

(5) *There is no digit which appears in all the integers.*

Answer: 32.
Solution: Associate with any a_i the set M_i of its digits. By (1), (2) and (3) the numbers are uniquely determined by their associated subsets of $\{1, 2, \ldots, 6\}$. By (4) the sets are intersecting. Partition the 64 subsets of $\{1, 2, \ldots, 6\}$ into 32 pairs of complementary sets $(X, \{1, 2, \ldots, 6\} - X)$. Obviously, at most one of the two sets in such a pair can be a M_i, since the two sets are non-intersecting. Hence, $n \leq 32$. Consider the 22 subsets with at least four elements and the 10 subsets with three elements containing 1. Hence, $n = 32$.

06.7. *A photographer took some pictures at a party with* 10 *people. Each of the* 45 *possible pairs of people appears together on exactly one photo, and each photo depicts two or three people. What is the smallest possible number of photos taken?*
Answer: 19.
Solution: Let x be the number of triplet photos (depicting three people, that is, three pairs) and let y be the number of pair photos (depicting two people, that is, one pair). Then $3x + y = 45$.

Each person appears with nine other people, and since 9 is odd, each person appears on at least one pair photo. Thus $y \geq 5$, so that $x \leq 13$. The total number of photos is $x + y = 45 - 2x \geq 45 - 2 \cdot 13 = 19$.

On the other hand, 19 photos will suffice. We number the persons $0, 1, \ldots, 9$, and will proceed to specify 13 triplet photos. We start with making triplets without common pairs of the persons 1–8:

$$123, 345, 567, 781$$

Think of the persons 1–8 as arranged in order around a circle. Then the persons in each triplet above are separated by at most one person. Next we make triplets containing 0, avoiding previously mentioned pairs by combining 0 with two people among the persons 1–8 separated by two persons:

$$014, 085, 027, 036$$

Then we make triplets containing 9, again avoiding previously mentioned pairs by combining 9 with the other four possibilities of two people among 1–8 being separated by two persons:

$$916, 925, 938, 947$$

Finally, we make our last triplet, again by combining people from 1–8: 246. Here 2 and 4, and 4 and 6, are separated by one person, but those pairs were not accounted for in the first list, whereas 2 and 6 are separated by three persons, and have not been paired before. We now have 13 photos of 39 pairs. The remaining 6 pairs appear on 6 pair photos.
Remark: This problem is equivalent to asking how many complete 3-graphs can be packed (without common edges) into a complete 10-graph.

06.8. *The director has found out that six conspiracies have been set up in his department, each of them involving exactly three persons. Prove that the director can split the department in two laboratories so that none of the conspirative groups is entirely in the same laboratory.*
Solution: Let the department consist of n persons. Clearly $n > 4$ (because $\binom{4}{3} < 6$). If $n = 5$, take three persons who do not make a conspiracy and put them in one laboratory,

the other two in another. If $n = 6$, note that $\binom{6}{3} = 20$, so we can find a three-person set such that neither it nor its complement is a conspiracy; this set will form one laboratory. If $n \geq 7$, use induction. We have $\binom{n}{2} \geq \binom{7}{2} = 21 > 6 \cdot 3$, so there are two persons A and B who are not together in any conspiracy. Replace A and B by a new person AB and use the inductive hypothesis; then replace AB by initial persons A and B.

06.9. *To every vertex of a regular pentagon a real number is assigned. We may perform the following operation repeatedly: we choose two adjacent vertices of the pentagon and replace each of the two numbers assigned to these vertices by their arithmetic mean. Is it always possible to obtain the position in which all five numbers are zeroes, given that in the initial position the sum of all five numbers is equal to zero?*

Answer: No.

Solution: We will show that starting from the numbers $-\frac{1}{5}, -\frac{1}{5}, -\frac{1}{5}, -\frac{1}{5}, \frac{4}{5}$ we cannot get five zeroes. By adding $\frac{1}{5}$ to all vertices we see that our task is equivalent to showing that beginning from numbers 0, 0, 0, 0, 1 and performing the same operations we can never get five numbers $\frac{1}{5}$. This we prove by noticing that in the initial position all the numbers are "binary rational" – that is, of the form $\frac{k}{2^m}$, where k is an integer and m is a non-negative integer – and an arithmetic mean of two binary rationals is also such a number, while the number $\frac{1}{5}$ is not of such form.

06.10. 162 *pluses and* 144 *minuses are placed in a* 30×30 *table in such a way that each row and each column contains at most* 17 *signs. (No cell contains more than one sign.) For every plus we count the number of minuses in its row and for every minus we count the number of pluses in its column. Find the maximum of the sum of these numbers.*

Answer: $1296 = 72 \cdot 18$.

Solution: In the statement of the problem there are two kinds of numbers: "horizontal" (that has been counted for pluses) and "vertical" (for minuses). We will show that the sum of numbers of each type reaches its maximum on the same configuration.

We restrict our attention to the horizontal numbers only. Consider an arbitrary row. Let it contains p pluses and m minuses, $m + p \leq 17$. Then the sum that has been counted for pluses in this row is equal to mp. Let us redistribute this sum between all signs in the row. More precisely, let us write the number $mp/(m + p)$ in every nonempty cell in the row. Now the whole "horizontal" sum equals to the sum of all 306 written numbers.

Now let us find the maximal possible contribution of each sign in this sum. That is, we ask about maximum of the expression $f(m, p) = mp/(m + p)$ where $m + p \leq 17$. Remark that $f(m, p)$ is an increasing function of m. Therefore if $m + p < 17$ then increasing of m will also increase the value of $f(m, p)$. Now if $m + p = 17$ then $f(m, p) = m(17 - m)/17$ and, obviously, it has maximum $72/17$ when $m = 8$ or $m = 9$.

So all the 306 summands in the horizontal sum will be maximal if we find a configuration in which every non-empty row contains 9 pluses and 8 minuses. The similar statement holds for the vertical sum. In order to obtain the desired configuration take a square 18×18 and draw pluses on 9 generalized diagonals and minuses on 8 other generalized diagonals (the 18th generalized diagonal remains empty).

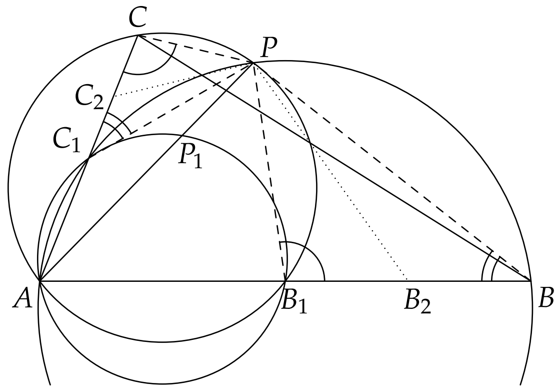
06.11. *The altitudes of a triangle are* 12, 15 *and* 20. *What is the area of the triangle?*

Answer: 150.

Solution: Denote the sides of the triangle by a, b and c and its altitudes by h_a, h_b and h_c. Then we know that $h_a = 12$, $h_b = 15$ and $h_c = 20$. By the well known relation $a : b = h_b : h_a$ it follows $b = \frac{h_a}{h_b} a = \frac{12}{15} a = \frac{4}{5} a$. Analogously, $c = \frac{h_a}{h_c} a = \frac{12}{20} a = \frac{3}{5} a$. Thus half of the triangle's circumference is $s = \frac{1}{2}(a + b + c) = \frac{1}{2}\left(a + \frac{4}{5} a + \frac{3}{5} a\right) = \frac{6}{5} a$. For the area Δ of the triangle we have $\Delta = \frac{1}{2} a h_a = \frac{1}{2} a\, 12 = 6a$, and also by the well known Heron formula $\Delta = \sqrt{s(s-a)(s-b)(s-c)} = \sqrt{\frac{6}{5}a \cdot \frac{1}{5}a \cdot \frac{2}{5}a \cdot \frac{3}{5}a} = \sqrt{\frac{6^2}{5^4}a^4} = \frac{6}{25}a^2$. Hence, $6a = \frac{6}{25}a^2$, and we get $a = 25$ ($b = 20$, $c = 15$) and consequently $\Delta = 150$.

06.12. *Let ABC be a triangle, let B_1 be the midpoint of the side AB and C_1 the midpoint of the side AC. Let P be the point of intersection, other than A, of the circumscribed circles around the triangles ABC_1 and AB_1C. Let P_1 be the point of intersection, other than A, of the line AP with the circumscribed circle around the triangle AB_1C_1. Prove that $2AP = 3AP_1$.*

Solution: Since $\angle PBB_1 = \angle PBA = 180° - \angle PC_1A = \angle PC_1C$ and $\angle PCC_1 = \angle PCA = 180° - \angle PB_1A = \angle PB_1B$ it follows that $\triangle PBB_1$ is similar to $\triangle PC_1C$. Let B_2 and C_2 be the midpoints of BB_1 and CC_1 respectively. It follows that $\angle BPB_2 = \angle C_1PC_2$ and hence $\angle B_2PC_2 = \angle BPC_1 = 180° - \angle BAC$, which implies that AB_2PC_2 lie on a circle. By similarity it is now clear that $AP/AP_1 = AB_2/AB_1 = AC_2/AC_1 = 3/2$.



06.13. *In a triangle ABC, points D, E lie on sides AB, AC respectively. The lines BE and CD intersect at F. Prove that if*

$$BC^2 = BD \cdot BA + CE \cdot CA,$$

then the points A, D, F, E lie on a circle.

Solution: Let G be a point on the segment BC determined by the condition $BG \cdot BC = BD \cdot BA$. (Such a point exists because $BD \cdot BA < BC^2$.) Then the points A, D, G, C lie on a circle. Moreover, we have

$$CE \cdot CA = BC^2 - BD \cdot BA = BC \cdot (BG + CG) - BC \cdot BG = CB \cdot CG,$$

hence the points A, B, G, E lie on a circle as well. Therefore

$$\angle DAG = \angle DCG, \qquad \angle EAG = \angle EBG,$$

which implies that

$$\begin{aligned}\angle DAE + \angle DFE &= \angle DAG + \angle EAG + \angle BFC \\ &= \angle DCG + \angle EBG + \angle BFC.\end{aligned}$$

But the sum on the right side is the sum of angles in $\triangle BFC$. Thus $\angle DAE + \angle DFE = 180°$, and the desired result follows.

06.14. *There are 2006 points marked on the surface of a sphere. Prove that the surface can be cut into 2006 congruent pieces so that each piece contains exactly one of these points inside it.*

Solution: Choose a North Pole and a South Pole so that no two points are on the same parallel and no point coincides with either pole. Draw parallels through each point.

Divide each of these parallels into 2006 equal arcs so that no point is the endpoint of any arc. In the sequel, "to connect two points" means to draw the smallest arc of the great circle passing through these points. Denote the points of division by $A_{i,j}$, where i is the number of the parallel counting from North to South ($i = 1, 2, \ldots, 2006$), and $A_{i,1}, A_{i,2}, \ldots, A_{i,2006}$ are the points of division on the i'th parallel, where the numbering is chosen such that the marked point on the i'th parallel lies between $A_{i,i}$ and $A_{i,i+1}$.

Consider the lines connecting gradually

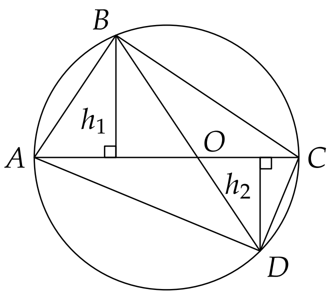
$$\begin{gathered} N - A_{1,1} - A_{2,1} - A_{3,1} - \cdots - A_{2006,1} - S \\ N - A_{1,2} - A_{2,2} - A_{3,2} - \cdots - A_{2006,2} - S \\ \vdots \\ N - A_{1,2006} - A_{2,2006} - A_{3,2006} - \cdots - A_{2006,2006} - S \end{gathered}$$

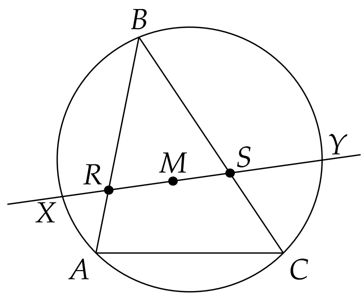
These lines divide the surface of the sphere into 2006 parts which are congruent by rotation; each part contains one of the given points.

06.15. *Let the medians of the triangle ABC intersect at the point M. A line t through M intersects the circumcircle of ABC at X and Y so that A and C lie on the same side of t. Prove that $BX \cdot BY = AX \cdot AY + CX \cdot CY$.*

Solution: Let us start with a lemma: If the diagonals of an inscribed quadrilateral $ABCD$ intersect at O, then $\frac{AB \cdot BC}{AD \cdot DC} = \frac{BO}{OD}$. Indeed,

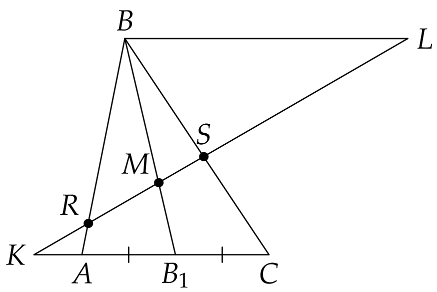
$$\frac{AB \cdot BC}{AD \cdot DC} = \frac{\frac{1}{2} AB \cdot BC \cdot \sin B}{\frac{1}{2} AD \cdot DC \cdot \sin D} = \frac{\text{area}(ABC)}{\text{area}(ADC)} = \frac{h_1}{h_2} = \frac{BO}{OD}.$$





Now we have (from the lemma) $\frac{AX \cdot AY}{BX \cdot BY} = \frac{AR}{RB}$ and $\frac{CX \cdot CY}{BX \cdot BY} = \frac{CS}{SB}$, so we have to prove $\frac{AR}{RB} + \frac{CS}{SB} = 1$.

Suppose at first that the line RS is not parallel to AC. Let RS intersect AC at K and the line parallel to AC through B at L. So $\frac{AR}{RB} = \frac{AK}{BL}$ and $\frac{CS}{SB} = \frac{CK}{BL}$; we must prove that $AK + CK = BL$. But $AK + CK = 2KB_1$, and $BL = \frac{BM}{MB_1} \cdot KB_1 = 2KB_1$, completing the proof.



If $RS \parallel AC$, the conclusion is trivial.

06.16. *Are there four distinct positive integers such that adding the product of any two of them to* 2006 *yields a perfect square?*

Answer: No, there are no such integers.

Solution: Suppose there are such integers. Let us consider the situation modulo 4. Then each square is 0 or 1. But $2006 \equiv 2 \pmod 4$. So the product of each two supposed numbers must be $2 \pmod 4$ or $3 \pmod 4$. From this it follows that there are at least three odd numbers (because the product of two even numbers is $0 \pmod 4$). Two of these odd numbers are congruent modulo 4, so their product is $1 \pmod 4$, which is a contradiction.

06.17. *Determine all positive integers n such that $3^n + 1$ is divisible by n^2.*

Answer: Only $n = 1$ satisfies the given condition.

Solution: First observe that if $n^2 \mid 3^n + 1$, then n must be odd, because if n is even, then 3^n is a square of an odd integer, hence $3^n + 1 \equiv 1 + 1 = 2 \pmod 4$, so $3^n + 1$ cannot be divisible by n^2 which is a multiple of 4.

Assume that for some $n > 1$ we have $n^2 \mid 3^n + 1$. Let p be the smallest prime divisor of n. We have shown that $p > 2$. It is also clear that $p \neq 3$, since $3^n + 1$ is never divisible by 3. Therefore $p \geq 5$. We have $p \mid 3^n + 1$, so also $p \mid 3^{2n} - 1$. Let k be the smallest positive integer such that $p \mid 3^k - 1$. Then we have $k \mid 2n$, but also $k \mid p - 1$ by Fermat's theorem. The numbers $3^1 - 1$, $3^2 - 1$ do not have prime divisors other than 2, so $p \geq 5$ implies $k \geq 3$. This means that $\gcd(2n, p-1) \geq k \geq 3$, and therefore $\gcd(n, p-1) > 1$, which contradicts the fact that p is the *smallest* prime divisor of n. This completes the proof.

06.18. *For a positive integer n let a_n denote the last digit of $n^{(n^n)}$. Prove that the sequence (a_n) is periodic and determine the length of the minimal period.*

Solution: Let b_n and c_n denote the last digit of n and n^n, respectively. Obviously, if $b_n = 0, 1, 5, 6$, then $c_n = 0, 1, 5, 6$ and $a_n = 0, 1, 5, 6$, respectively.

If $b_n = 9$, then $n^n \equiv 1 \pmod 2$ and consequently $a_n = 9$. If $b_n = 4$, then $n^n \equiv 0 \pmod 2$ and consequently $a_n = 6$.

If $b_n = 2, 3, 7,$ or 8, then the last digits of n^m run through the periods: $2 - 4 - 8 - 6$, $3 - 9 - 7 - 1$, $7 - 9 - 3 - 1$ or $8 - 4 - 2 - 6$, respectively. If $b_n = 2$ or $b_n = 8$, then $n^n \equiv 0 \pmod 4$ and $a_n = 6$.

In the remaining cases $b_n = 3$ or $b_n = 7$, if $n \equiv \pm 1 \pmod 4$, then so is n^n.

If $b_n = 3$, then $n \equiv 3 \pmod{20}$ or $n \equiv 13 \pmod{20}$ and $n^n \equiv 7 \pmod{20}$ or $n^n \equiv 13 \pmod{20}$, so $a_n = 7$ or $a_n = 3$, respectively.

If $b_n = 7$, then $n \equiv 7 \pmod{20}$ or $n \equiv 17 \pmod{20}$ and $n^n \equiv 3 \pmod{20}$ or $n^n \equiv 17 \pmod{20}$, so $a_n = 3$ or $a_n = 7$, respectively.

Finally, we conclude that the sequence (a_n) has the following period of length 20:

$$1 - 6 - 7 - 6 - 5 - 6 - 3 - 6 - 9 - 0 - 1 - 6 - 3 - 6 - 5 - 6 - 7 - 6 - 9 - 0$$

06.19. *Does there exist a sequence $a_1, a_2, a_3, \dots$ of positive integers such that the sum of every n consecutive elements is divisible by n^2 for every positive integer n?*

Answer: Yes. One such sequence begins 1, 3, 5, 55, 561, 851, 63253, 110055,. . ..

Solution: We will show that whenever we have positive integers $a_1, \dots, a_k$ such that $n^2 \mid a_{i+1} + \dots + a_{i+n}$ for every $n \leq k$ and $i \leq k - n$, then it is possible to choose a_{k+1} such that $n^2 \mid a_{i+1} + \dots + a_{i+n}$ for every $n \leq k + 1$ and $i \leq k + 1 - n$. This directly implies the positive answer to the problem because we can start constructing the sequence from any single positive integer.

To obtain the necessary property, it is sufficient for a_{k+1} to satisfy

$$a_{k+1} \equiv -(a_{k-n+2} + \cdots + a_k) \pmod{n^2}$$

for every $n \leq k+1$. This is a system of $k+1$ congruences.

Note first that, for any prime p and positive integer l such that $p^l \leq k+1$, if the congruence with module p^{2l} is satisfied then also the congruence with module $p^{2(l-1)}$ is satisfied. To see this, group the last p^l elements of $a_1, \ldots, a_{k+1}$ into p groups of p^{l-1} consecutive elements. By choice of $a_1, \ldots, a_k$, the sums computed for the first $p-1$ groups are all divisible by $p^{2(l-1)}$. By assumption, the sum of the elements in all p groups is divisible by p^{2l}. Hence the sum of the remaining p^{l-1} elements, that is $a_{k-p^{l-1}+2} + \cdots + a_{k+1}$, is divisible by $p^{2(l-1)}$.

Secondly, note that, for any relatively prime positive integers c, d such that $cd \leq k+1$, if the congruences both with module c^2 and module d^2 hold then also the congruence with module $(cd)^2$ holds. To see this, group the last cd elements of $a_1, \ldots, a_{k+1}$ into d groups of c consecutive elements, as well as into c groups of d consecutive elements. Using the choice of $a_1, \ldots, a_k$ and the assumption together, we get that the sum of the last cd elements of $a_1, \ldots, a_{k+1}$ is divisible by both c^2 and d^2. Hence this sum is divisible by $(cd)^2$.

The two observations let us reject all congruences except for the ones with module being the square of a prime power p^l such that $p^{l+1} > k+1$. The resulting system has pairwise relatively prime modules and hence possesses a solution by the Chinese Remainder Theorem.

06.20. *A* 12*-digit positive integer consisting only of digits* 1, 5 *and* 9 *is divisible by* 37*. Prove that the sum of its digits is not equal to* 76*.*

Solution: Let N be the initial number. Assume that its digit sum is equal to 76.

The key observation is that $3 \cdot 37 = 111$, and therefore $27 \cdot 37 = 999$. Thus we have a divisibility test similar to the one for divisibility by 9: for $x = a_n 10^{3n} + a_{n-1}10^{3(n-1)} + \cdots + a_1 10^3 + a_0$, we have $x \equiv a_n + a_{n-1} + \cdots + a_0 \pmod{37}$. In other words, if we take the digits of x in groups of three and sum these groups, we obtain a number congruent to x modulo 37.

The observation also implies that $A = 111\,111\,111\,111$ is divisible by 37. Therefore the number $N - A$ is divisible by 37, and since it consists of the digits 0, 4 and 8, it is divisible by 4. The sum of the digits of $N - A$ equals $76 - 12 = 64$. Therefore the number $\frac{1}{4}(N - A)$ contains only the digits 0, 1, 2; it is divisible by 37; and its digits sum up to 16. Applying our divisibility test to this number, we sum four three-digit groups consisting of the digits 0, 1, 2 only. No digits will be carried, and each digit of the sum S is at most 8. Also S is divisible by 37, and its digits sum up to 16. Since $S \equiv 16 \equiv 1 \pmod{3}$ and $37 \equiv 1 \pmod{3}$, we have $S/37 \equiv 1 \pmod{3}$. Therefore $S = 37(3k+1)$, that is, S is one of 037, 148, 259, 370, 481, 592, 703, 814, 925; but each of these either contains the digit 9 or does not have a digit sum of 16.

RESULTS

In the the tables below, the following abbreviations have been used.

Bel	Belgium	Ger	Germany	Pol	Poland
Blr	Belarus	Ice	Iceland	StP	St. Petersburg
Den	Denmark	Lat	Latvia	Swe	Sweden
Est	Estonia	Lit	Lithuania		
Fin	Finland	Nor	Norway		

Belgium and Belarus participated in 2005 and 2004, respectively.

2002	Algebra					Combinatorics					Geometry					Number theory					**Total**
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	
Den	5	5	0	1	4	5	0	0	5	5	0	5	0	0	2	0	0	2	0	0	39
Est	5	5	3	0	5	4	4	0	5	0	0	4	5	0	2	2	0	4	5	0	53
Fin	5	5	2	5	5	5	5	0	5	5	4	1	1	0	5	1	2	0	0	1	57
Ger	5	5	1	4	5	3	0	5	0	0	5	5	5	1	5	0	5	2	4	5	65
Ice	5	2	2	0	3	5	2	0	0	2	5	0	1	0	0	3	0	0	0	0	30
Lat	5	5	2	0	5	5	5	0	0	5	0	0	1	1	2	1	1	4	0	1	43
Lit	5	5	5	4	5	5	0	0	5	5	5	1	5	1	0	2	0	5	5	5	68
Nor	4	5	1	4	4	5	5	2	1	5	5	5	5	0	5	0	5	5	0	5	71
Pol	5	5	0	5	5	5	0	0	5	4	0	5	5	0	0	4	3	5	5	5	66
StP	5	5	5	5	5	5	1	5	0	5	5	5	5	5	0	5	5	5	5	5	86
Swe	0	5	1	0	5	5	0	5	0	5	1	5	2	0	5	1	4	2	0	1	47
Mean	4.5	4.7	2.0	2.5	4.6	4.7	2.0	1.5	2.4	3.7	2.7	3.3	3.2	0.7	2.4	1.7	2.3	3.1	2.2	2.5	

2003	Algebra					Combinatorics					Geometry					Number theory					**Total**
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	
Den	5	5	0	0	1	0	1	5	5	0	0	5	0	5	0	2	0	1	3	5	43
Est	3	5	5	5	4	0	1	5	5	1	0	2	5	0	5	5	5	0	5	0	61
Fin	1	5	5	5	5	0	1	0	5	0	0	0	0	0	0	5	0	1	5	0	38
Ger	1	0	0	4	2	0	1	1	5	0	1	5	0	0	0	4	5	0	3	0	32
Ice	1	0	5	2	1	0	1	0	0	0	0	5	0	0	5	5	0	0	5	1	31
Lat	5	5	5	5	2	5	4	0	2	0	0	0	0	0	0	5	5	0	5	5	53
Lit	1	5	3	5	5	0	5	0	1	0	0	5	0	0	5	5	3	0	4	2	49
Nor	1	5	0	0	5	0	3	2	5	1	0	5	0	0	0	5	5	0	2	0	39
Pol	5	5	5	5	5	0	5	2	4	0	0	5	5	0	5	5	5	0	5	2	68
StP	5	5	5	5	5	5	5	5	0	5	0	5	5	5	5	5	5	0	5	5	85
Swe	0	0	5	5	5	0	4	0	0	0	2	1	0	0	0	5	2	0	5	3	37
Mean	2.5	3.6	3.5	3.7	3.6	0.9	2.8	1.8	2.9	0.6	0.3	3.5	1.4	0.9	2.3	4.6	3.2	0.2	4.3	2.1	

2004	Algebra					Number theory					Combinatorics					Geometry					**Total**
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	
Blr	5	4	5	5	5	5	5	5	5	5	2	4	5	1	3	5	5	5	0	0	79
Den	0	5	0	5	5	5	5	0	0	0	2	5	5	2	5	1	5	5	0	0	55
Est	1	1	5	0	5	5	5	0	5	5	5	5	5	1	5	5	5	5	5	5	78
Fin	0	5	5	5	3	5	2	0	5	4	1	4	5	0	5	5	5	5	4	5	73
Ger	4	5	0	5	5	5	5	5	5	2	2	4	5	1	1	0	5	0	0	0	59
Ice	4	5	0	5	0	5	3	1	2	0	0	0	5	0	3	1	4	5	1	5	49
Lat	4	5	5	3	5	5	0	0	5	4	2	1	5	0	5	0	5	5	0	0	59
Lit	5	4	4	5	5	5	5	0	0	2	3	0	2	0	5	0	5	5	5	5	65
Nor	0	5	0	5	5	5	4	3	5	4	5	3	0	5	5	0	3	5	0	5	67
Pol	5	4	5	0	5	5	5	3	5	5	0	4	5	1	5	5	5	5	5	5	82
StP	5	4	5	4	5	5	5	5	5	5	5	5	5	1	5	5	5	5	5	0	89
Swe	1	4	0	3	0	5	5	0	0	2	2	0	0	0	5	0	5	0	0	0	32
Mean	2.8	4.2	2.8	3.8	4.0	5.0	4.1	1.8	3.5	3.2	2.4	2.9	3.9	1.0	4.3	2.2	4.8	4.2	2.1	2.5	

2005	Algebra					Combinatorics					Geometry					Number theory					**Total**
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	
Bel	1	5	0	1	0	1	3	4	0	5	4	2	3	0	0	0	0	0	5	5	39
Den	0	0	0	5	0	0	0	1	0	5	0	0	5	0	2	0	0	1	0	5	24
Est	0	0	0	2	5	5	0	1	5	5	0	1	3	0	5	0	0	0	0	5	37
Fin	5	5	0	5	0	5	0	5	5	5	5	5	5	5	5	0	0	5	5	5	75
Ger	1	5	0	5	0	5	1	1	4	5	5	5	2	0	0	5	3	5	5	5	62
Ice	1	0	0	0	5	0	0	1	1	5	5	0	4	0	0	0	0	3	0	5	30
Lat	4	0	0	5	1	0	4	0	1	5	5	5	5	5	0	0	5	5	5	5	60
Lit	5	5	0	4	0	0	0	5	5	5	0	0	5	0	5	0	0	4	5	5	53
Nor	3	5	0	4	0	1	2	4	4	5	0	0	3	2	0	0	5	0	5	5	48
Pol	5	5	0	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	95
StP	5	5	0	5	0	0	0	5	5	5	5	5	2	5	5	5	5	5	1	5	73
Swe	5	0	0	5	0	0	0	0	5	5	5	5	2	0	0	0	0	5	5	5	47
Mean	2.9	2.9	0.0	3.8	1.3	1.8	1.2	2.7	3.3	5.0	3.2	2.8	3.7	1.8	2.2	1.2	1.9	3.2	3.4	5.0	

2006	Algebra					Combinatorics					Geometry					Number theory					**Total**
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	
Den	5	0	0	1	3	3	5	0	5	0	0	2	0	0	0	0	1	5	0	5	35
Est	5	4	5	0	0	5	5	0	0	0	5	0	5	0	5	0	5	3	0	3	50
Fin	5	5	5	1	0	5	3	0	0	0	1	5	0	0	5	5	5	5	0	0	50
Ger	5	0	3	0	0	5	3	4	0	3	5	5	0	0	5	5	1	5	5	5	59
Ice	5	5	0	0	0	3	1	0	0	3	0	0	1	2	0	0	0	5	0	0	25
Lat	1	0	2	0	0	2	0	0	0	0	2	0	5	0	5	5	5	5	5	5	42
Lit	5	5	5	1	3	5	5	5	5	1	5	5	0	0	5	5	1	3	2	0	66
Nor	5	5	5	4	1	5	5	0	5	0	5	1	0	0	0	5	1	5	1	0	53
Pol	5	3	5	4	3	4	5	0	3	0	5	5	5	5	5	5	0	5	5	5	77
StP	5	5	5	4	5	5	5	5	5	5	5	5	4	0	5	5	5	5	5	5	93
Swe	5	5	5	5	1	5	1	2	0	0	5	5	0	0	5	0	4	3	0	5	56
Mean	4.6	3.4	3.6	1.8	1.5	4.3	3.5	1.5	2.1	1.1	3.5	3.0	1.8	0.6	3.6	3.2	2.5	4.5	2.1	3.0	

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